The Order of Large Random Permutations with Cycle Weights

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Summary

The field of non-uniform random permutations has grown rapidly in recent years, particularly due to its relevance in mathematical biology and theoretical physics. Ercolani and Ueltschi recently considered a model of random permutations of n objects which appears in the study of quantum Bose gas in statistical mechanics: each cycle of length m is assigned an individual weight $\theta_m \geq 0$. In this thesis we investigate random permutations under two different choices of parameters θ_m: the so-called generalized Ewens parameters and a measure with polynomial cycle weights $\theta_m = m^\gamma$, $\gamma > 0$.

The crucial tool to study uniform random permutations is the so-called Feller coupling. Unfortunately, due to a lack of compatibility between the different dimensions, the Feller coupling is no longer available for the weighted measure. Therefore, new approaches are needed to examine the behavior of weighted permutations. Combining tools from combinatorics and complex analysis such as singularity analysis and saddle-point analysis, we extend some classical results of uniform random permutations to our setting and also establish properties of the order of random permutations which are new even for the uniform measure.

The order $O_n(\sigma)$ of a permutation σ is the smallest integer $k \geq 1$ such that the k-th iterate of σ gives the identity. The most famous result about the order of a uniformly chosen permutation is due to Erdős and Turán who showed in 1965 that $\log O_n(\sigma)$ satisfies a central limit theorem. In Chapter 4 we establish a variety of properties of $O_n(\sigma)$ for the generalized Ewens measure. The extension of the Erdős-Turán law to this model is straightforward. Furthermore, we obtain a local limit theorem as well as, under some extra moment condition, a precise large deviations estimate. We also provide a precise expression of the expected value of $\log O_n(\sigma)$, which has an immediate interpretation in terms of the Riemann hypothesis.

Let $\mathrm{d}_b(n)$ denote the total variation distance of the process which counts the cycles of size $1, 2, \ldots, b$ and a process $(Z_1, Z_2, \ldots, Z_b)$ of independent Poisson random variables. It is well-known that for uniform random permutations $\mathrm{d}_b(n) \to 0$ as $n \to \infty$ if and only if $b = o(n)$. This approximation by independent random variables allows one to prove many asymptotic properties of the cycle count process. This convergence also holds for the generalized Ewens measure, but its extension to permutations with polynomial cycle weights is more intricate. Using saddle-point analysis, we prove in Chapter 5 that $\mathrm{d}_b(n) \to 0$ as $n \to \infty$ if and only if $b = o(n^{\frac{1}{1+\gamma}})$. This condition is more restrictive than in the uniform setting. It will turn out that only the behavior of the small cycles can be controlled by the independent approximating random variables. However, we show that this total variation estimate is useful to extend the Erdős-Turán law to random permutations with polynomial cycle weights. Moreover, we prove a precise large deviations estimate for $\log O_n$.

Zusammenfassung

Das Interesse in nicht-gleichverteilte zufällige Permutationen ist stetig gewachsen in den vergangenen Jahren, insbesondere aufgrund ihrer vielfältigen Anwendungen in der theoretischen Physik und mathematischen Biologie. Ercolani und Ueltschi betrachteten kürzlich ein Modell von zufälligen Permutationen von n Objekten, das im Zusammenhang mit Bosegas in der statistischen Mechanik auftritt: jedem Zykel der Länge m wird ein individuelles Gewicht $\theta_m \geq 0$ zugeordnet. In der vorliegenden Dissertation untersuchen wir zufällige Permutationen bezüglich zwei Arten von Parametern θ_m: zum einen die sogenannten verallgemeinerten Ewens-Parameter und zum anderen polynomielle Parameter $\theta_m = m^\gamma$, $\gamma > 0$.

Eine wichtige Methode für die Untersuchung von gleichverteilten zufälligen Permutationen ist das sogenannte Feller-coupling. Aufgrund von fehlender Kompatibilität zwischen den verschiedenen Dimensionen ist das Feller-coupling nicht verfügbar für gewichtete Permutationen. Wir kombinieren Elemente der Kombinatorik und der komplexen Analysis, um einerseits klassische Resultate von gleichverteilten Permutationen auf unser Modell zu übertragen und um andererseits neue Eigenschaften der Ordnung einer Permutation zu etablieren.

Die Ordnung $O_n(\sigma)$ einer Permutation σ ist definiert als die kleinste natürliche Zahl k sodass die k-malige Iteration von σ die identische Permutation ergibt. Das bedeutendste Resultat über $O_n(\sigma)$ beruht auf Erdős und Turán: sie zeigten 1965, dass $\log O_n(\sigma)$ einer uniform gewählten Permutation σ annähernd normalverteilt ist. In Kapitel 4 weisen wir eine Vielzahl von Eigenschaften von $O_n(\sigma)$ bezüglich der verallgemeinerten Ewens-Parameter nach. Zunächst zeigen wir, dass der Satz von Erdős und Turán auch für dieses Modell erfüllt ist. Weiterhin beweisen wir einen lokalen Grenzwertsatz, große Abweichungen und einen präzisen Ausdruck für den Erwartungswert von $\log O_n(\sigma)$, von dem sich eine Äquivalenz zur Riemannschen Vermutung ableiten lässt.

Bezeichne $\mathrm{d}_b(n)$ den Abstand in Totalvariation von dem Prozess $(C_1, C_2, \ldots, C_b)$, der die Zykel der Länge $1, 2, \ldots, b$ zählt, und einem Prozess von unabhängigen Poisson-Verteilungen $(Z_1, Z_2, \ldots, Z_b)$. Für gleichverteilte Permutationen ist bekannt, dass $\mathrm{d}_b(n) \to 0$ mit $n \to \infty$ genau dann, wenn $b = o(n)$. Mit Hilfe dieser Approximation wurden zahlreiche asymptotische Eigenschaften von $C_1, C_2, \ldots$ bewiesen. Diese Konvergenz ist auch erfüllt für verallgemeinerte Ewens-Parameter, jedoch ist die Erweiterung für polynomielle Gewichte komplizierter. Mit Hilfe von Sattelpunkt-Analysis zeigen wir in Kapitel 5, dass $\mathrm{d}_b(n) \to 0$ mit $n \to \infty$ genau dann, wenn $b = o(n^{\frac{1}{1+\gamma}})$. Diese Bedingung ist stärker als die oben genannte. Tatsächlich können in unserem Modell nur die kleinen Zykel direkt mit den unanhängigen approximierenden Zufallsvariablen kontrolliert werden. Trotzdem können wir zeigen, wie diese Konvergenz in Totalvariation genutzt werden kann, um den Satz von Erdős und Turán auf Permutationen mit polynomiellen Zykelgewichten zu übertragen. Weiterhin beweisen wir ein Prinzip der großen Abweichungen für $\log O_n$.

Contents

1

Setting the scene

A hundred prisoners, each uniquely identified by a number between 1 *and* 100*, have been sentenced to death. The director of the prison gives them a last chance. He has a cabinet with* 100 *drawers numbered* 1 *to* 100*. In each, he'll place at random a card with a prisoner's number. Prisoners will be allowed to enter the room one after the other and open, then close again,* 50 *drawers of their own choosing, but will not be allowed to communicate with the other prisoners. The goal of each prisoner is to locate the drawer that contains his own number. If all prisoners succeed, then they will all be spared; if at least one fails, they will all be executed.*
There are two mathematicians among the prisoners. The first one, a pessimist, declares that their overall chances of success are only of order $1/2^{100} = 8 \cdot 10^{-31}$*. The second one, a combinatorialist, claims that he has strategy which has a greater than* 30% *chance of success. Who is right? (This problem is borrowed from [39, II.15]. The solution can be found at the end of Section 1.1.)*

This type of problem can be formulated in terms of random permutations, which have been studied for many decades and have become increasingly relevant in the recent years. Apart from its rich mathematical structure, the interest in random permutations is justified by its wide range of applications in mathematical biology [33, 42, 50, 71] and mathematical physics [15, 17, 30, 63]. In this thesis, we focus on non-uniform random permutations with cycle weights which were introduced recently in the works of Betz et al. [18] and Ercolani and Ueltschi [30]. We combine tools from combinatorics and complex analysis to extend some classical results on uniform permutations to this model. Furthermore, we provide a variety of results on the order of weighted random permutations. It is natural to first prove a central limit theorem for our setting which generalizes the famous Erdős-Turán law. In fact, we are able to prove a finer convergence, namely mod-φ convergence, a notion which was recently introduced by Kowalski and Nikeghbali, together with other coauthors

[26, 47, 54, 53]. This allows us to establish results such as a local limit theorem and large deviations estimates which are new even for the uniform measure.

We now give a short description of the layout of this thesis. In the remainder of the present chapter we give some background about random permutations and their cycle structure as well as an overview of results on weighted random permutations. In Chapter 2 we explain the underlying methods which form the basis of our study: singularity analysis and saddle-point analysis on the one hand and mod-φ convergence on the other hand. Using these techniques, we extend in Chapter 3 some general results on uniform permutations to our model. In particular, we examine the behavior of large cycles and provide a functional central limit theorem. The most substantial results in this thesis are contained in Chapters 4 and 5, where we present a comprehensive study of the order of weighted random permutations.

1.1 Random permutations and their cycle structure

We denote by $\mathfrak{S}_n$ the symmetric group, that is the set of all permutations on n letters. Every permutation $\sigma \in \mathfrak{S}_n$ can be written as a product of disjoint cycles $\sigma = \sigma_1\sigma_2 \ldots \sigma_l$, where we denote by λ_i the length of cycle σ_i. Then define the *cycle counts* C_m to be the number of cycles of length m in the decomposition of σ, that is

$$C_m := C_m^{(n)}(\sigma) := \#\{i : \lambda_i = m\}. \tag{1.1}$$

Clearly, the cycle counts satisfy

$$n = \sum_{m=1}^{n} m\, C_m. \tag{1.2}$$

If a permutation is chosen uniformly at random from the $n!$ choices in $\mathfrak{S}_n$, then the cycle counts are dependent random variables.

The crucial tool to study random permutations under the uniform measure is the *Feller coupling*, which was introduced by Feller [34] in 1945 to establish a central limit theorem for the number of cycles. The idea is that a random permutation on n objects can be constructed from a sequence of n independent Bernoulli random variables such that the steps between successive 1's in the sequence correspond to the cycle lengths in the permutation. To be more precise, let $X_1, X_2, \ldots$ be a sequence of independent Bernoulli random variables with

$$\mathbb{P}[X_m = 1] = 1 - \mathbb{P}[X_m = 0] = \frac{1}{m}, \qquad m \geq 1. \tag{1.3}$$

For each n, one can construct a permutation $\sigma \in \mathfrak{S}_n$ recursively using the sequence $X_n, X_{n-1}, \ldots, X_1$ as follows. Considering the canonical cycle notation for σ, one always starts with "(1" in the first cycle. If $X_n = 1$ we close off the cycle and start the next with the smallest available integer, that is we get "(1)(2". If $X_n = 0$, choose randomly one of the remaining $n - 1$ integers and place it to the right of 1 in the same cycle, so that we get "$(1, j$" with $j \in \{2, 3, \ldots, n-1\}$. Continue with $X_{n-1}, X_{n-2}, \ldots$ to produce a permutation on n objects with cycles ordered by their smallest integer. Remarkably, the distribution of σ is uniform on $\mathfrak{S}_n$. Furthermore, define

$$Z_{m,l} = \sum_{i=l+1}^{\infty} X_i(1 - X_{i+1}) \cdots (1 - X_{i+m-1}) X_{i+m}$$

and $Z_m = Z_{m,0}$, then

$$C_m^{(n)} = Z_m - Z_{m,n-j} + X_{n-m+1}(1 - X_{n-m+2}) \cdots (1 - X_n).$$

Another remarkable fact is that the Z_m's are independent Poisson random variables with respective means $1/m$.

With this construction at hand, it is easy to show that the process of cycle counts converges in distribution to independent Poisson random variables on $\mathbb{N}$. In other terms,

$$(C_1^{(n)}, C_2^{(n)}, \ldots) \xrightarrow{d} (Z_1, Z_2, \ldots) \qquad \text{as } n \to \infty, \tag{1.4}$$

where $\xrightarrow{d}$ denotes convergence in distribution and the Z_m denote independent Poisson random variables with mean $1/m$. This result has been first established by Goncharov [41] in 1944. David and Barton [24] proved in 1962 that the rate of convergence to the Poisson distribution is super-exponential in n. An important feature of the cycle count process is that it can be described in terms of these independent Poisson random variables conditioned on the value of a weighted sum. Define

$$\mathcal{Z}_{0n} = \sum_{m=1}^{n} m\, Z_m.$$

Then the following so-called *conditioning relation* holds:

$$\mathcal{L}(C_1^{(n)}, C_2^{(n)}, \ldots, C_n^{(n)}) = \mathcal{L}(Z_1, Z_2, \ldots, Z_n \mid \mathcal{Z}_{0n} = n). \tag{1.5}$$

For many purposes (1.4) is not strong enough to deduce properties of the random permutations from the independent limiting process, because (1.4) only involves the convergence of the distribution of $(C_1, C_2, \ldots, C_b)$ for fixed b as $n \to \infty$. However, many natural properties of the cycle count process jointly depend on all components, even though the contribution of the larger ones is less relevant. Therefore, estimates were needed for b and n growing simultaneously. For $1 \le b \le n$, denote by

$$\mathrm{d}_b(n) := \mathrm{d}_{\mathrm{TV}}\left(\mathcal{L}(C_1, C_2, \ldots, C_b), \mathcal{L}(Z_1, Z_2, \ldots, Z_b)\right) \tag{1.6}$$

the *total variation distance* of the distributions of $(C_1, C_2, \ldots, C_b)$ and $(Z_1, Z_2, \ldots, Z_b)$. In a discrete probability space, the convergence in (1.4) is equivalent to $\mathrm{d}_b(n) \to 0$

for all b fixed as $n \to \infty$. By means of the Feller coupling and the conditioning relation (1.5), Arratia et al. [7] showed that, as $n \to \infty$,

$$\mathrm{d}_b(n) \to 0 \quad \text{if and only if} \quad b = o(n). \tag{1.7}$$

This proves to be a very powerful tool which is often used to establish properties of complicated functionals of the cycle counts via an approximation based on independent Poisson random variables. For more details on total variation distance estimates see Section 5.2.

A comprehensive summary of classical results on uniform random permutations can be found in [4, Chapter 1]. Here we present merely those which will be of further interest in this thesis. First, we define the *total number of cycles*

$$T_n := T_{0n} := \sum_{m=1}^{n} C_m. \tag{1.8}$$

Several authors have studied the asymptotic distribution of T_n. Using moment generating functions, Goncharov [41] proved a central limit theorem: as $n \to \infty$, the convergence

$$\frac{T_n - \log(n)}{\sqrt{\log(n)}} \xrightarrow{d} \mathcal{N}(0,1) \tag{1.9}$$

holds, where $\mathcal{N}(0,1)$ denotes the standard normal distribution. Alternative proofs were given later; see [34, 66, 52]. The previous result can be generalized to a functional central limit theorem describing the distribution of the cycles of size not exceeding n^x for $0 \leq x \leq 1$. Define

$$W_n(x) := \frac{\sum_{m=1}^{\lfloor n^x \rfloor} C_m - x\log(n)}{\sqrt{\log(n)}}.$$

DeLaurentis and Pittel [25] proved that, as $n \to \infty$ and for $0 \leq x \leq 1$, the process W_n converges weakly to the standard Brownian motion on $[0,1]$.

Another quantity of interest is the *order* of a permutation, denoted by $O_n = O_n(\sigma)$, which is defined as the smallest integer $k \geq 1$ such that the k-th iterate of σ is the identity. Landau [55] proved in 1909 that the maximum of the order of all $\sigma \in \mathfrak{S}_n$ satisfies, for $n \to \infty$, the asymptotic

$$\max_{\sigma \in \mathfrak{S}_n} \log O_n(\sigma) \sim \sqrt{n\log(n)}. \tag{1.10}$$

On the other hand, $O_n(\sigma)$ can be computed as the least common multiple of the cycle lengths of σ. Thus, if σ is a permutation that consists of only one cycle of length n, then $\log O_n(\sigma) = \log(n)$, and $(n-1)!$ of all $n!$ permutations share this property. Considering these two extremal types of behavior, the famous result of Erdős and Turán [32] seems even more remarkable: they showed in 1965 that a uniformly chosen random permutation satisfies, as $n \to \infty$, the central limit theorem

$$\frac{\log O_n - \frac{1}{2}\log^2(n)}{\sqrt{\frac{1}{3}\log^3(n)}} \xrightarrow{d} \mathcal{N}(0,1). \tag{1.11}$$

The original proof was direct and rather technical. Thereafter, several authors gave probabilistic proofs of this result, among them Best [14] in 1970, DeLaurentis and Pittel [25] in 1985 who use a functional central limit theorem for the cycle counts, and Arratia and Tavaré [9] in 1992, whose proof is based on the Feller coupling. For more details on the order of random permutations we refer the reader to Chapters 4 and 5 where we present a variety of properties of $\log O_n$ in a more general setting.

The first proofs of these and other limit theorems were rather technical and often involved complicated analytic methods. With estimates like (1.7) which allow to decouple the small cycle components into independent Poisson random variables, much more elementary proofs became available. The crucial point is to show that for an appropriate choice of b, the cycles of size greater than b have a negligible contribution to the distribution of the functional in question. See [9] and the references therein for more details.

In contrast to (1.4), which can be interpreted as a result on small cycles, or (1.9) and (1.11), which are results on the whole cycle count process, one can also study large cycles. Denote by $\ell^{(r)}(\sigma)$ the length of the r-th longest cycle in a permutation $\sigma \in \mathfrak{S}_n$; we set $\ell^{(r)} = 0$ if the permutation has fewer than r cycles. Goncharov [41] proved in 1944 that

$$n^{-1}\,\ell^{(1)} \xrightarrow{d} L_1 \qquad \text{as } n \to \infty, \tag{1.12}$$

where the distribution of L_1 is determined by the Dickman function ρ, introduced by Dickman [27] to describe the largest prime factor of an integer. Later, Kingman [50] and Shmidt and Vershik [68] showed in 1977 that, as $n \to \infty$,

$$\left(\frac{\ell^{(1)}}{n}, \frac{\ell^{(2)}}{n}, \dots\right) \xrightarrow{d} (L_1, L_2, \dots), \tag{1.13}$$

where the vector on the right-hand side has a distribution known as the Poisson-Dirichlet distribution with parameter 1. We refer to Section 3.4 for further results on large cycles.

The connection between permutations and prime numbers that emerges in the previous statement does not appear accidentally. In fact, it is only one of many examples describing the similar behavior of permutations (decomposed into cycles) and integers (factorized into prime numbers). The object of the following section is to shed light on the structural similarity between the two settings.

This is the solution to the prisoners problem stated at the beginning of this chapter; see [39, III.10]. The better strategy goes as follows. Each prisoner first opens the *drawer which corresponds to his number. If his number is not there, he uses the number he just found to access another drawer, where he finds another number that directs him to a third drawer, and so on, hoping to return to his original drawer in at most* 50 *trials (the last opened drawer will then contain his number). For each prisoner the probability of success is* $1/2$*, as one expects. The magic arises from the fact that the events of the different prisoners success are highly correlated.*

This strategy globally succeeds provided the initial permutation σ *defined by the number* σ_i *that is assigned to drawer* i *has no cycles of length greater than* 50*. Let us compute the probability of this event. The number of permutations of* 100 *letters with a cycle of length at least* m *is*

$$\binom{100}{m}(m-1)!\,(100-m)! = \frac{100!}{m}$$

and thus a uniform permutation has a cycle of length at least m *with probability* $1/m$*. We conclude*

$$\mathbb{P}[\textit{Each cycle has length at most } 50] = 1 - \sum_{m=51}^{100} \frac{1}{m} \approx 0.31$$

and thus it is indeed approximately 31%*.*

1.2 The connection between permutations and prime numbers

While studying properties of permutations, one frequently discovers a connection with prime numbers, such as for the behavior of large cycles which was presented in the previous section. A structural analogy of the factorization of permutations into cycles and integers into prime numbers was first observed by Knuth and Trabb Pardo [51]. Many examples illustrating this profound similarity are available; see [4, Chapter 1] for a comprehensive overview.

In Chapters 4 and 5 in this thesis we establish a variety of properties of the order of random permutations and it will turn out that many of our proofs involve prime numbers. The discussion which follows gives an informal motivation for the appearance of prime numbers in our calculations; however, it is not required for the proofs and may be omitted if desired.

Recall that the cycle counts $C_1(\sigma), C_2(\sigma), \ldots, C_n(\sigma)$ of a permutation $\sigma \in \mathfrak{S}_n$ satisfy (1.2). On the other hand, any integer n decomposes uniquely as a product of primes: denote by $p_1, p_2 \ldots, p_n$ the first n prime numbers, then there exists non-negative numbers $\alpha_1^{(n)}, \ldots, \alpha_n^{(n)}$ such that

$$n = \prod_{m=1}^{n} p_m^{\alpha_m^{(n)}} \quad \text{and thus} \quad \log(n) = \sum_{m=1}^{n} \alpha_m^{(n)} \log(p_m). \tag{1.14}$$

Let us compare the frequency of the prime numbers with the frequency of permutations which consist of only one cycle. Informally speaking, the *prime number theorem* states that one out of every $\log(n)$ integers up to n is a prime number. On the other hand, there are $(n-1)! = |\mathfrak{S}_n|/n$ cycles of length n in $\mathfrak{S}_n$, meaning that one out of n permutations of n elements consists of only one cycle. Consequently, the scale n in the setting of permutations should be replaced by a scale $\log(n)$ when considering the setting of integers.

Indeed, this adjustment seems natural when comparing (1.2) with (1.14). Furthermore, it also appears when translating the central limit theorem (1.9) for the total cycle number into the language of natural numbers. Let $\omega(n)$ denote distinct prime divisors of a natural number n, then the Erdős-Kác theorem [31] states that, as $n \to \infty$,

$$\frac{\omega(n) - \log\log(n)}{\sqrt{\log\log(n)}} \xrightarrow{d} \mathcal{N}(0,1). \tag{1.15}$$

The connection becomes even more striking when studying large cycles. For fixed $u > 0$, the probability that the largest cycle of a uniform permutation on n letters contains no more than n/u elements is approximately $\rho(u)$ for large n, where ρ denotes the Dickman function; see (1.12). Translated in terms of integers and replacing again n by $\log(n)$, the question is how often some function of the largest prime factor p of n is at most $\log(n)/u$. This function should be $\log(p)$, thus $\log(p) \leq \log(n)/u$ if and only if $p \leq n^{1/u}$. Indeed, it was proved by Dickman [27] that the probability that the largest prime factor of an integer $k \leq n$ is at most $n^{1/u}$ is approximately $\rho(u)$ for large n.

There is a variety of further examples illustrating the structural analogy of permutations and integers, but to justify the comparability of the two systems it is necessary to explain *why* they are similar. Arratia et al. [5] explain this phenomenon with the fact that the prime factorization and the cycle decomposition share two important features. The cycle counts satisfy the conditioning relation (1.5) and a very close relative for the prime factorization does also hold. This is a rather algebraic property while the second one is essentially analytic: for fixed x, the number of components (cycles or primes) of size at most x has a limiting distribution as $n \to \infty$, and the mean of this limit is asymptotically equal to $\vartheta \log(x)$ as $x \to \infty$, for some $\vartheta > 0$. This is the so-called *logarithmic condition* and for uniform permutations (recall (1.4)) we have indeed

$$\sum_{m \leq x} \mathbb{E}[Z_m] = \sum_{m \leq x} \frac{1}{m} \sim \log(x).$$

On the other hand, choose a random integer and recall the prime factorization (1.14), then the multiplicities $\alpha_m^{(n)}$ of the prime factors p_m are random variables and it is well-known that

$$(\alpha_1^{(n)}, \alpha_2^{(n)}, \dots) \xrightarrow{d} (Z_1, Z_2, \dots) \qquad \text{as } n \to \infty,$$

where the Z_m are independent and geometrically distributed with parameter $1/m$ (see [19] for an explicit proof). Then replacing as above x by $\log(x)$ and summing over the prime numbers gives (see [45])

$$\sum_{\log(p)\leq\log(x)} \mathbb{E}[Z_p] = \sum_{p\leq x} \frac{1}{p} \sim \log\log(x).$$

The similarity of the prime factorization and the cycle decomposition based on these two essential properties led to new conjectures in number theory. It would go beyond the scope of this thesis to provide more details; we refer to [3] and the references therein for further reading.

1.3 Weighted random permutations

A natural generalization of the previous setting is to choose a permutation according to a biased probability measure rather than selecting each one uniformly. Therefore, let $\vartheta > 0$ be a constant and define the so-called *Ewens measure* as

$$\mathbb{P}_\vartheta[\sigma] := \frac{1}{h_n n!} \prod_{m=1}^{n} \vartheta^{C_m} = \frac{\vartheta^{T_n}}{h_n n!}, \tag{1.16}$$

where the total number of cycles T_n is defined as in (1.8) and $h_n = h_n(\vartheta)$ is a normalization constant that makes $\mathbb{P}_\vartheta$ a probability distribution (set $h_0 := 1$). This formula was derived in 1972 by Ewens [33] in the context of population genetics where C_m is the number of alleles represented m times in a sample of n genes, and in 1974 by Antoniak [1] in a Bayesian nonparametric statistics setting. Notice that the choice $\vartheta = 1$ gives the uniform distribution on $\mathfrak{S}_n$.

The Feller coupling described in Section 1.1 is still available under the Ewens measure, but instead of (1.3) we have

$$\mathbb{P}[X_m = 1] = 1 - \mathbb{P}[X_m = 0] = \frac{\vartheta}{\vartheta + m - 1}, \qquad m \geq 1.$$

The essential features of uniform permutations, such as (1.4), (1.5) and (1.7), remain valid under the Ewens measure when the independent Poisson variables Z_m are such that $\mathbb{E}[Z_m] = \vartheta/m$; see [7]. Thus, the classical limit theorems still hold true but the parameter ϑ appears in the rescaling. For example (1.9) becomes

$$\frac{T_n - \vartheta\log(n)}{\sqrt{\vartheta\log(n)}} \xrightarrow{d} \mathcal{N}(0,1) \tag{1.17}$$

as $n \to \infty$; see [9]. A functional version of this result was proved by Hansen [43] and Donelly et al. [28]. The analog of the Erdős-Turán law (1.11) for the Ewens measure is

$$\frac{\log O_n - \frac{\vartheta}{2}\log^2(n)}{\sqrt{\frac{\vartheta}{3}\log^3(n)}} \xrightarrow{d} \mathcal{N}(0,1) \tag{1.18}$$

as $n \to \infty$, and was proven in [9]. For a detailed account on results about permutations under the Ewens measure see [4] and the references therein. Models with a two-parameter version of (1.16) have been introduced by Pitman [64] and further studied by Feng and Hoppe [35].

The literature on non-uniform permutations has grown quickly in recent years, particularly due to its relevance in mathematical biology and theoretical physics. In this thesis, we focus on random permutations with cycle weights as introduced in the recent works of Betz et al. [18] and Ercolani and Ueltschi [30]. In their model, each cycle of length m is assigned an individual weight $\theta_m \geq 0$. More precisely, the probability of $\sigma \in \mathfrak{S}_n$ is defined as follows:

Definition 1.1. *Let $\Theta = (\theta_m)_{m\geq 1}$ be given, with $\theta_m \geq 0$ for every $m \geq 1$. Then define the weighted probability of a permutation $\sigma \in \mathfrak{S}_n$ as*

$$\mathbb{P}_\Theta[\sigma] := \frac{1}{h_n n!} \prod_{m=1}^{n} \theta_m^{C_m} \tag{1.19}$$

with $h_n = h_n(\Theta)$ the partition function that makes $\mathbb{P}_\Theta$ a probability distribution, $h_0 := 1$.

Notice that the choice $\theta_m \equiv \vartheta$ gives the Ewens measure defined in (1.16). Benaych-Georges [13] and Timashev [73] considered a version of this model involving parameters $\theta_m \in \{0, 1\}$ with finitely many 1's. Another setting of interest is given when the asymptotic behavior of the parameter θ_m is fixed for large m. Such a model was studied by Barbour and Granovsky [11] who assumed that the parameters are converging and satisfy some regular variation condition.

A variety of different regimes of parameters that are of interest for the study of quantum gas in statistical mechanics and that have a connection with the Bose-Einstein condensation, have been considered by Betz and Ueltschi together with other coauthors in a series of papers [15, 16, 17, 18, 30]. In their model, the parameters $\Theta = (\theta_m)_{m\geq 1}$ may depend on quantities such as the temperature, the density or the particle interaction and thus they do not necessarily take a simple form.

The challenging point is that due to a lack of compatibility between the different dimensions, the Feller coupling is no longer available for the measure $\mathbb{P}_\Theta$. Therefore, new approaches are needed to generalize the classical results on uniform random permutations to the weighted measure. The crucial feature of $\mathbb{P}_\Theta$ is that it is invariant on conjugacy classes. Using generating series and complex analysis methods, several natural properties of weighted random permutations were recently obtained by several authors. The starting point of the study is the generating series

$$g_\Theta(t) := \sum_{m=1}^{\infty} \frac{\theta_m}{m} t^m. \tag{1.20}$$

As we shall see, the asymptotic behavior of random variables with respect to the weighted measure $\mathbb{P}_\Theta$ strongly depends on the analytic properties of the function

g_Θ. According to the nature of the parameters $\Theta = (\theta_m)_{m\geq 1}$, the right method to investigate this function is either singularity analysis (when g_Θ exhibits moderate growth at its dominant singularity) or saddle-point analysis (when g_Θ exhibits some form of exponential growth at its dominant singularity). The basic ideas of both methods are described in Section 2.2. Now, we give an overview of existing results obtained with these methods.

To get a first intuition on the influence of different classes of parameters on the cycle structure of a permutation, it is useful to consider the length of the cycle that contains the index 1, denoted by $\mathcal{L}_1$, which may be interpreted as the length of a typical cycle. Given the structure of (1.19) one might expect that for increasing parameters θ_m, longer cycles are more likely. This turns out to be wrong; see [30, Table 1]. In the uniform case, $\mathbb{P}(\mathcal{L}_1 = \ell) = 1/n$ for all $\ell = 1, 2, \ldots$ and thus a typical cycle has length of order n. For parameters $\theta_m = m^\gamma$ with $\gamma > 0$, a typical cycle has length of order $n^{\frac{1}{1+\gamma}}$ but it has length of order $(\log(n))^{1/\gamma}$ for parameters $\theta_m = e^{m^\gamma}$ with $0 < \gamma < 1$. This surprising behavior is better understood from the perspective of statistical mechanics; see [17, 18, 30]. Apart from the typical cycle length, Ercolani and Ueltschi [30] prove for several classes of parameters (namely those satisfying sub-exponential decay or growth, algebraic growth or that are asymptotically Ewens) the analogous result of (1.4), that is

$$(C_1^{(n)}, C_2^{(n)}, \ldots) \xrightarrow{d} (Z_1, Z_2, \ldots) \qquad \text{as } n \to \infty, \tag{1.21}$$

where the Z_m are independent Poisson random variables with $\mathbb{E}[Z_m] = \theta_m/m$. Furthermore, they prove estimates for the expected value of the total number of cycles $T_n = \sum_{m=1}^{n} C_m$. For uniform permutations, $\mathbb{E}[T_n]$ is of order $\log(n)$ (see (1.9)) and the same is true for asymptotically Ewens parameters (see [30, Theorem 6.1]), while for parameters $\theta_m = m^\gamma$ with $\gamma > 0$, the expected value is of order $n^{\frac{1}{1+\gamma}}$; see [30, Theorem 5.1]. They apply a refined saddle-point analysis of generating functions which is a very general method that allows them to obtain results for a large variety of parameters. However, they do not get information on the rates of convergence or any central limit theorem.

As a complementary perspective, Nikeghbali and Zeindler [60] determine a class of parameters $\Theta = (\theta_m)_{m\geq 1}$ via analytic properties of its generating function g_Θ defined as in (1.20) which define the so-called *generalized Ewens measure*. Roughly speaking, this measure comprises all types of parameters $\Theta = (\theta_m)_{m\geq 1}$ such that g_Θ exhibits some logarithmic singularities. Under this measure, (1.21) holds (see [60, Corollary 3.2]) as well as an analogous version of (1.17) (see [60, Theorem 4.3]). Furthermore, they get estimates on the rates of convergence and large deviations estimates for the total number of cycles T_n.

Apart from the generalized Ewens measure, another class of parameters of special interest are the polynomial parameters $\theta_m = m^\gamma$ for some $\gamma > 0$. As mentioned before, under these parameters (1.21) holds and estimates for the expectation of T_n were proved in [30, Theorem 5.1]. Assuming the so-called *log-admissibility* on the generating series (1.20), Maples et al. [57, Theorem 1.1] established a central limit theorem for T_n analogous to (1.17). They also get large deviations estimates for T_n; see [57, Theorem 4.2]. Further results on this class of parameters where obtained by Cipriani and Zeindler [21], who study the limit shape of Young diagrams associated with random permutations.

1.4 Overview of main results

In this thesis we investigate random permutations with respect to two different weighted measures; the generalized Ewens measure and a measure with polynomial weights.

Chapters 3 and 4 are devoted to the generalized Ewens measure. In Chapter 3 we establish results that were obtained jointly with Nikeghbali and Zeindler; see [59]. In particular, we examine the behavior of large cycles and provide a functional central limit theorem for the total cycle number.

The most substantial results of this thesis are contained in Chapters 4 and 5, where we present a comprehensive study of the order of a permutation with respect to the generalized Ewens measure (see Chapter 4) and for a measure with polynomial parameters $\theta_m = m^\gamma$, $\gamma > 0$ (see Chapter 5). These results were obtained in collaboration with Zeindler; see [70, 69].

A_n-permutations under the generalized Ewens measure

The first objects of our interest are the so-called *A-permutations*, which are permutations that can be decomposed into cycles whose lengths are all in a set A. They have been intensively studied for the uniform and Ewens measure. We extend some classical results to the generalized Ewens measure, which in particular allows to consider A_n-permutations, where the sets A_n depend on the degree n of $\mathfrak{S}_n$.

In Section 3.4 we show that the size ordered cycle lengths converge in law to a Poisson-Dirichlet distribution. More precisely, denote by $\ell^{(1)}(\sigma)$ the length of the longest cycle of a permutation σ, by $\ell^{(2)}(\sigma)$ the length of the second longest cycle and so forth. Then, under some mild extra condition on the generating function g_Θ as in (1.20), we show that, as $n \to \infty$,

$$\left(\frac{\ell^{(1)}}{n}, \frac{\ell^{(2)}}{n}, \dots\right) \xrightarrow{d} \mathcal{PD}(\vartheta),$$

where $\mathcal{PD}(\vartheta)$ denotes the Poisson-Dirichlet distribution with parameter ϑ. This agrees with the results of Shmidt and Vershik [68] and Kingman [50], who studied the same asymptotic behavior with respect to the Ewens measure; see (1.13).

Furthermore, in Section 3.5 we consider the number of cycles in a permutation with lengths not exceeding n^x with $0 \le x \le 1$ and show that this process converges, after normalization, to a standard Brownian motion:

$$\frac{\sum_{m=1}^{\lfloor n^x \rfloor} C_m - x\vartheta \log(n)}{\sqrt{\vartheta \log(n)}} \xrightarrow{d} \mathcal{W}(x), \tag{1.22}$$

as $n \to \infty$, where $\mathcal{W}$ denotes a standard Brownian motion on $[0, 1]$. This extends the results of DeLaurentis and Pittel [25] (uniform measure) and Hansen [43] (Ewens measure) to the A_n-weighted measure.

Our method is a combination of tools from combinatorics and complex analysis. Its advantage is that it is very flexible and it allows to study the quantity of interest under further restrictions, for example random variables that only involve cycles with even or odd cycle length; see Section 3.6.

The order of permutations under the generalized Ewens measure

We establish a variety properties of the order of a permutation for the generalized Ewens measure. The extension of the Erdős-Turán law (1.18) to this model is straightforward; see Section 4.1. Apart from precise estimates for the expected value of $\log O_n$ and rates of convergence for the Erdős-Turán law or functional versions of it, until recently not much was known about the behavior of $\log O_n$ even for the uniform measure.

In Section 4.3, we obtain a local limit theorem for $\log O_n$. To this end, we define

$$\Omega_n := \frac{\log O_n - \frac{\vartheta}{2}\log^2(n)}{\log^{4/3}(n)} \qquad \text{and} \qquad \sigma_n := \sqrt{\frac{\vartheta}{3}}\log^{1/6}(n).$$

Under some mild extra conditions on the parameters $\Theta = (\theta_m)_{m \geq 1}$, we will show that for any bounded Borel subset $B \subset \mathbb{R}$ with boundary of Lebesgue measure zero

$$\lim_{n\to\infty} \sigma_n \, \mathbb{P}_\Theta \left[\Omega_n \in B\right] = \frac{m(B)}{\sqrt{2\pi}}$$

holds, where $m(B)$ denotes the Lebesgue measure of B.

Furthermore, we obtain precise large deviations estimates. For Ω_n and λ_n as above, we prove in Section 4.4 that under some extra moment condition the following holds for any $x > 0$:

$$\mathbb{P}_\Theta\left[\Omega_n \geq x\,\sigma_n^2\right] = \frac{\exp(-\sigma_n^2 \frac{x^2}{2} + \frac{x^3\vartheta}{18})}{\sigma_n x \sqrt{2\pi}}\,(1 + o(1)).$$

In Section 4.6 we give a precise estimate for the expected value of $\log O_n$, which extends results of Zacharovas [78]. We require additional assumptions on the generating series g_Θ defined in (1.20). Then

$$\begin{aligned}\mathbb{E}_\Theta\left[\log O_n\right] &= \frac{\vartheta}{2}\log^2(n) - \vartheta\log(n)\big(1 - \log(\vartheta\log(n))\big)\\ &\quad + \sum_{\varrho}\Gamma(-\varrho)(\vartheta\log(n))^{\varrho} + \mathcal{O}\left((\log\log(n))^3\right),\end{aligned}$$

where $\sum_\varrho$ denotes the sum over the non-trivial zeros of the Riemann zeta function. In particular, we will show that a certain behavior of this expansion is equivalent to the Riemann hypothesis; see Corollary 4.27.

Total variation asymptotic for permutations with polynomial cycle weights

In Chapter 5 polynomial parameters $\theta_m = m^\gamma, \gamma > 0$, are considered. For this model it is known that (1.21) holds. Recall that $\mathrm{d}_b(n)$ defined in (1.6) denotes the total variation distance of the cycle count process and independent Poisson random variables Z_m. We will prove in Section 5.2 that for appropriately chosen Z_m the following holds:

$$\mathrm{d}_b(n) \to 0 \quad \text{if and only if} \quad b = o(n^{\frac{1}{1+\gamma}}) \tag{1.23}$$

and we also obtain a rate of convergence; see Theorem 5.1.

Comparing (1.7) and (1.23) we notice that for polynomial parameters, the cycle counts exhibit a more dependent structure. An intuitive explanation is the following. In the Ewens case, a typical cycle has length of order n, and the numbers of cycles of length $o(n)$ are asymptotically independent. For polynomial parameters $\theta_m = m^\gamma$, a typical cycle has length of order $n^{\frac{1}{1+\gamma}}$ (see [30, Theorem 5.1]), providing an intuitive justification for the bound on b in (1.23).

For the Ewens measure, several applications demonstrating the power of (1.7) are known; see [8] for a detailed account. The condition $b = o(n^{\frac{1}{1+\gamma}})$ in the present model is more restrictive than the condition $b = o(n)$ for the Ewens measure. It will turn out that only the behavior of the small cycles can be controlled with (1.23), since in many cases the cycles with length slightly longer than $n^{\frac{1}{1+\gamma}}$ have a non-negligible contribution to the behavior of quantities involving the whole cycle count process.

The order of permutations with polynomial cycle weights

As for the generalized Ewens measure, we will establish several results on the order of weighted permutations for parameters $\theta_m = m^\gamma, \gamma > 0$. It is natural to begin with an Erdős-Turán law analogous to (1.18) for this model. Indeed, in Section 5.3 we will prove that for $0 < \gamma < 1$, as $n \to \infty$,

$$\frac{\log O_n - G(n)}{\sqrt{F(n)}} \xrightarrow{d} \mathcal{N}(0,1),$$

where $G(n) = \mathcal{O}\big(n^{\frac{\gamma}{1+\gamma}} \log(n)\big)$ and $F(n) = \mathcal{O}\big(n^{\frac{\gamma}{1+\gamma}} \log^2(n)\big)$; see Theorem 5.11 for the precise statement.

Furthermore, though the bound in (1.23) is too small to investigate the whole cycle count process via the independent Poisson random variables, we will show in

Section 5.3 how (1.23) can be used to study the small components by proving a functional version of the Erdős-Turán law. For $x > 0$ define $x^* := \lfloor x\, n^{\frac{\gamma}{1+\gamma}} \rfloor$ and

$$B_n(x) := \frac{\log O_{x^*} - \frac{1}{1+\gamma}\, x^\gamma \log(n)\, n^{\frac{\gamma^2}{1+\gamma}}}{\frac{\sqrt{\gamma}}{1+\gamma} \log(n)\, n^{\frac{\gamma^2}{2(1+\gamma)}}}, \tag{1.24}$$

where $O_{x^*}(\sigma) := \operatorname{lcm}\{m \leq x^*; C_m > 0\}$. We will show that for $0 < \gamma < 1$, the process $B_n(x)$ converges weakly to $\mathcal{W}(x^\gamma)$ as $n \to \infty$, where $\mathcal{W}$ denotes a standard Brownian motion.

Moreover, we prove a precise large deviations estimate for $\log O_n$. Define

$$\Omega_n := \frac{\log O_n - \lambda_n \log(n)(1+\gamma)^{-2}}{\lambda_n^{1/3} \log(n)(1+\gamma)^{-2}},$$

where λ_n is a parameter of order $n^{\frac{\gamma}{1+\gamma}}$ (see the precise statement in Theorem 5.2). Then, for $0 < \gamma < 1$ and for any $x > 0$ the following asymptotic holds:

$$\mathbb{P}_\Theta\left[\Omega_n \geq x\lambda_n^{1/3}\right] = \frac{\exp(-\lambda_n^{1/3}\frac{x^2}{2} + \frac{x^3}{6})}{x\lambda_n^{1/6}\sqrt{2\pi}}\,(1 + o(1)).$$

2

Overview of methods

This chapter presents an overview of methods and techniques which will be relevant to establish our results. Section 2.1 is devoted to some basic facts about the symmetric group $\mathfrak{S}_n$, partitions and generating functions, which are central concepts in combinatorial theory. In particular, we recall the important *cycle index theorem* which links generating functions with averages over $\mathfrak{S}_n$.

As we shall see throughout this thesis, the generating function of many quantities of interest will be known explicitly; however, there are no simple expression for the coefficients. Crucial to our study is the correspondence between the asymptotic expansion of the coefficients of a function and the asymptotic expansion of the generating function near its singularities. This is where methods of complex analysis such as *singularity analysis* and *saddle-point analysis* come into play; they are briefly illustrated in Section 2.2.

It turns out that several sequences of random variables under consideration converge in the so-called *mod-Gaussian* or *mod-Poisson* sense, a notion which was introduced in [47]. The motivation is as follows. When a sequence of random variables converges in distribution, then the corresponding sequence of characteristic functions possesses a limit in the sense of pointwise convergence. However, even if a sequence does not converge in distribution, its characteristic functions may decay precisely like those of a suitable law φ such as Gaussian or Poisson distributions. Such sequences are called mod-φ convergent. Section 2.3 provides an outline of properties and implications of this type of convergence, for example a local limit theorem and precise large deviations estimates.

2.1 The symmetric group & generating functions

All probability measures and functions considered in this thesis are invariant under conjugation and it is well known that the conjugation classes of $\mathfrak{S}_n$ can be parametrized by partitions of n. This can be seen as follows: let $\sigma \in \mathfrak{S}_n$ be an arbitrary permutation and write $\sigma = \sigma_1 \cdots \sigma_\ell$ with σ_i disjoint cycles of length λ_i. Since disjoint cycles commute, we can assume that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell$. Then $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ is a partition of n since $\sum_{m=1}^{\ell} \lambda_m = n$. We call this partition *cycle-type* of σ and $\ell = \ell(\lambda)$ its *length*. Then two elements $\sigma, \tau \in \mathfrak{S}_n$ are conjugate if and only if σ and τ have the same cycle-type; more details can be found for instance in [56]. For $\sigma \in \mathfrak{S}_n$ with cycle-type λ, we define C_m to be the number of cycles of size m, that is

$$C_m := \# \{i \, : \, \lambda_i = m\}. \tag{2.1}$$

Recall that u is a class function when it satisfies $u(\sigma) = u(\tau^{-1}\sigma\tau)$ for all $\sigma, \tau \in \mathfrak{S}_n$. It will turn out that all expectations of interest have the form $\frac{1}{n!}\sum_{\sigma \in \mathfrak{S}_n} u(\sigma)$ for a certain class function u. Since u is constant on conjugacy classes, it is more natural to sum over all conjugacy classes. This is subject of the following lemma.

Lemma 2.1. *Let $u : \mathfrak{S}_n \to \mathbb{C}$ be a class function, C_m be as in* (2.1) *and $\mathcal{C}_\lambda$ the conjugacy class corresponding to the partition λ. We then have*

$$\frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} u(\sigma) = \sum_{\lambda \vdash n} \frac{1}{z_\lambda} u(\mathcal{C}_\lambda)$$

with $z_\lambda := \prod_{m=1}^{n} m^{C_m} C_m!$ and $\sum_{\lambda \vdash n}$ the sum over all partitions of n.

Given a sequence $(a_n)_{n \in \mathbb{N}}$ of numbers, one can encode important information about this sequence into a power series called the generating series.

Definition 2.2. *Let $(a_m)_{m \in \mathbb{N}}$ be a sequence of complex numbers. We then define the generating function of $(a_m)_{m \in \mathbb{N}}$ as the formal power series*

$$g(t) = \sum_{m=0}^{\infty} a_m t^m. \tag{2.2}$$

We define $[t^m]\,[g(t)]$ to be the coefficient of t^m of $g(t)$, that is $[t^m]\,[g(t)] := a_m$.

As already mentioned in the previous chapter, a special generating function constructed with the coefficients $\Theta = (\theta_m)_{m \geq 1}$ given in Definition 1.1 plays a crucial role in this thesis, namely

$$g_\Theta(t) := \sum_{m=1}^{\infty} \frac{\theta_m}{m} t^m. \tag{2.3}$$

As we will see, the asymptotic behavior of all random variables on the symmetric group $\mathfrak{S}_n$ with respect to the weighted measure $\mathbb{P}_\Theta$ strongly depends on analytic properties of this function.

The reason why generating functions are relevant is the possibility of identifying them without knowing the coefficients a_m explicitly. The basic idea is to apply tools from analysis to extract information about a_m, for large m, from the generating function. The following well-known identity is a special case of the general *Pólya's enumeration theorem* [65] and is sometimes called *cycle index theorem.* It links generating functions and averages over $\mathfrak{S}_n$.

Lemma 2.3. *Let $(a_m)_{m\in\mathbb{N}}$ be a sequence of complex numbers. Then*

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}\sum_{\sigma\in\mathfrak{S}_n}\prod_{m=1}^{\infty}a_m^{C_m}=\sum_{\lambda}\frac{1}{z_\lambda}\left(\prod_{m=1}^{\infty}(a_m t^m)^{C_m}\right)=\exp\left(\sum_{m=1}^{\infty}\frac{a_m}{m}t^m\right)$$

with the same z_λ as in Lemma 2.1. If one of the sums above is absolutely convergent, then so are the others.

Proof. The proof can be found in [56]. These identities can also be directly verified using the definitions of z_λ and the exponential function. The last statement follows from the dominated convergence theorem. □

The previous lemma provides a connection between the function g_Θ and the generating series with coefficients h_n.

Corollary 2.4. *Let g_Θ be the generating function as in* (2.3) *and let h_n be as in Definition 1.1. Then the following holds*

$$\sum_{n=0}^{\infty}h_n t^n=\exp(g_\Theta(t)). \tag{2.4}$$

Proof. This follows immediately from the definition of h_n in Definition 1.1 together with Lemma 2.3. □

The generating function (2.4) yields expressions for the factorial moments of the cycle counts.

Lemma 2.5. *We have for all $m,k\in\mathbb{N}$,*

$$\mathbb{E}_\Theta\left[(C_m)_k\right]=\left(\frac{\theta_m}{m}\right)^k\frac{h_{n-mk}}{h_n},$$

where $(c)_k := c(c-1)\cdots(c-k+1)$ denotes the Pochhammer symbol. Furthermore, for $m_1\neq m_2$,

$$\mathbb{E}_\Theta\left[C_{m_1}C_{m_2}\right]=\frac{\theta_{m_1}}{m_1}\frac{\theta_{m_2}}{m_2}\frac{h_{n-m_1-m_2}}{h_n}.$$

Proof. Recall Lemma 2.3 and set $a_m=\theta_m$, then differentiate the sum k times with respect to θ_m and obtain

$$\sum_{n=0}^{\infty}h_n\mathbb{E}_\Theta\left[(C_m)_k\right]t^n=\sum_{n=0}^{\infty}\frac{t^n}{n!}\sum_{\sigma\in\mathfrak{S}_n}(C_m)_k\prod_{m=1}^{\infty}\theta_m^{C_m}=\left(\frac{\theta_m}{m}t^m\right)^k\exp(g_\Theta(t)). \tag{2.5}$$

Identifying the coefficients of t^n on both sides completes the proof of the first assertion in Lemma 2.5. The proof of the second one is similar. □

Remark 2.6. It is now easy to see that under the mild condition $h_{n-1}/h_n \to r$, the convergence (1.4) holds with $\mathbb{E}_\Theta[Z_m] = \theta_m r^m/m$; see for instance [30, Corollary 2.3].

Typically, Lemma 2.5 is relevant in situations where one can express the quantity of interest in terms of the factorial moments of C_m. However, in our case it proves simpler to take a different approach, which was in particular applied by Hansen [43]. Assume for $t > 0$ that $G_\Theta(t) := \exp\bigl(g_\Theta(t)\bigr) < \infty$ with $g_\Theta(t)$ as in Corollary 2.4. Then set

$$\Omega_t := \dot{\bigcup}_{n\in\mathbb{N}} \mathfrak{S}_n$$

and define for $\sigma \in \mathfrak{S}_n$

$$\mathbb{P}_\Theta^t\left[\sigma\right] := \frac{1}{G_\Theta(t)} \frac{t^n}{n!} \prod_{m=1}^{n} \theta_m^{C_m}.$$

Lemma 2.3 shows that $\mathbb{P}_\Theta^t$ defines a probability measure on Ω_t. Furthermore, under $\mathbb{P}_\Theta^t$ the C_m are independent and Poisson distributed. To avoid confusion, we will write Z_m instead of C_m when we consider the measure $\mathbb{P}_\Theta^t$, that is the Z_m are independent Poisson random variables with $\mathbb{E}_\Theta^t\left[Z_m\right] = \theta_m t^m/m$. This follows easily with a calculation similar to the proof of Lemma 2.5. Moreover, the following conditioning relation holds:

$$\mathbb{P}_\Theta^t\left[\,\cdot \mid \mathfrak{S}_n\right] = \mathbb{P}_\Theta^n\left[\,\cdot\,\right]. \tag{2.6}$$

We also have

$$\mathbb{P}_\Theta^t\left[\mathfrak{S}_n\right] = t^n h_n \exp\bigl(-g_\Theta(t)\bigr),$$

which follows immediately from the definition of h_n in (1.19). Then the law of total probability yields

Lemma 2.7. *Let $t > 0$ be given so that $G_\Theta(t) < 0$. Suppose that $\Psi : \Omega_t \to \mathbb{C}$ is a random variable with $\mathbb{E}_\Theta^t\left[|\Psi|\right] < \infty$ and that Ψ only depends on the cycle counts, i.e. $\Psi = \Psi(C_1, C_2, \dots)$. We then have with $\Psi_n := \Psi\Big|_{\mathfrak{S}_n}$*

$$\exp\bigl(g_\Theta(t)\bigr)\mathbb{E}_\Theta^t\left[\Psi\right] = \sum_{n=1}^{\infty} h_n \mathbb{E}_\Theta^n\left[\Psi_n\right] t^n + \Psi(0).$$

The previous equation is stated only for a fixed t, but if both sides are complex analytic functions in t, then the equation is also valid as formal power series. If one chooses for instance $\Psi = (Z_m)_k$, one has $\mathbb{E}_\Theta^t\left[\Psi\right] = (\theta_m/m)^k t^{mk}$ and thus obtains (2.5).

We call this approach the *randomization method* and we will apply it several times in this thesis. To establish the functional central limit theorem stated in (1.22), we will have to show the tightness of a certain stochastic process and our argument is based on the randomization method. Moreover, for the study of properties of the order of a permutation in Chapter 4 and 5, we will need the generating series of the random variable

$$\log Y_n := \sum_{m=1}^{n} C_m \log(m).$$

With the above convention, we then have on Ω_t

$$\log Y_n = \sum_{m=1}^{n} Z_m \log(m).$$

Since $Z_1, Z_2, \ldots, Z_n$ are independent Poisson random variables with respective parameters $\theta_m t^m/m$, we obtain

$$\mathbb{E}_{\Theta}^{t}\left[e^{s\log Y_n}\right] = \mathbb{E}_{\Theta}^{t}\left[e^{s\sum_{m=1}^{n} Z_m \log(m)}\right] = \exp\Big(\sum_{m=1}^{n}\frac{\theta_m}{m}t^m(e^{s\log(m)}-1)\Big)$$

and then Lemma 2.7 yields

$$\sum_{n=0}^{\infty} h_n \mathbb{E}_{\Theta}[\exp(s\log Y_n)]t^n = \exp\left(\sum_{m=1}^{\infty}\frac{\theta_m}{m^{1-s}}t^m\right). \tag{2.7}$$

Furthermore, we will apply the randomization method in Section 5.2, where we compare the distribution of the cycle counts C_m with the distribution of the independent random variables Z_m. Notice that (2.6) implies the so-called *Conditioning Relation*

$$\mathcal{L}\big((C_1,\ldots,C_n)\big) = \mathcal{L}\Big((Z_1,\ldots,Z_n)|\sum_{k=1}^{n} kZ_k = n\Big).$$

This important relation is the key to the proof of the asymptotic (1.23).

2.2 Complex analysis methods

Generating functions as in Definition 2.2 are a central concept in combinatorial theory. In many interesting situations, the generating function is known explicitly, but no simple expression is available for its coefficients. However, it turns out that the singularities of a function provide valuable information about its coefficients. In particular, their asymptotic rate of growth depends only on *local properties* of the generating function, namely its dominant singularities. *Cauchy's residue theorem* relates the local properties of a function with its global behavior. An important application is *Cauchy's integral formula*, which expresses coefficients of analytic functions as contour integrals and thus leads to estimates of coefficients by adequately selecting the contours of integration.

For easy reference we revise in this section basic facts about complex analysis methods, such as singularity analysis and saddle-point asymptotics, which will be crucial to establish our results. The reader familiar with the theory is invited to skip directly to Section 2.3. For detailed account of this theory we refer to [39, Part B].

Recall that if a function g defined on an open connected subset Ω of the complex plane is analytic at $0 \in \Omega$, it can be expressed for all t in an open disc centered at 0 and contained in Ω as a convergent power series

$$g(t) = \sum_{n=0}^{\infty} a_n t^n. \tag{2.8}$$

Furthermore, there exists a disc (of possibly infinite radius) such that the series representing $g(t)$ is convergent for t inside the disc and divergent for t outside the disc. This is the so-called disc of convergence and its radius is the *radius of convergence* which we usually denote by ρ. The following simple lemma known as *Pringsheim's theorem* (see e.g. [39, Theorem IV.6]) is important in asymptotic enumeration when generating functions with non-negative coefficients are considered.

Lemma 2.8 (Pringsheim's theorem)**.** *Assume that $a_n \geq 0$ for every $n \geq 0$ and let the series expansion* (2.8) *have a finite radius of convergence ρ. Then the point $t = \rho$ is a singularity of the function g.*

Singularities of a function analytic at the origin which lie on the boundary of the disc of convergence are called *dominant singularities*, and they convey important information regarding the rate at which the coefficients grow. Pringsheim's theorem simplifies the search for dominant singularities of combinatorial generating functions; it is sufficient to examine the analyticity along the positive real line.

A key feature of integral calculus for analytic functions is that integrals are independent of the integration contour. This enables us to relate local characteristics of a function (such as its residues at poles) with global properties (its integral along closed curves). This is the subject of Cauchy's residue theorem; see e.g. [39, Theorem IV.3]. Let us first recall the definition of meromorphic functions and residues.

Definition 2.9 ([39], Definition IV.3)**.** *A function h is called meromorphic at* 0 *if for $t \neq 0$ in a neighborhood of* 0*, it can be represented as $g(t)/f(t)$ with g and f being analytic in* 0*. In that case, it admits near* 0 *an expansion of the form*

$$h(t) = \sum_{n \geq -M} h_n t^n.$$

If $h_{-M} \neq 0$ and $M \geq 1$, then h is said to have a pole of order M at $t = 0$. The coefficient h_{-1} is called the residue of h at $t = 0$ and is written as

$$\mathrm{Res}[h(t); t = 0].$$

Theorem 2.10 (Cauchy's residue theorem)**.** *Let h be meromorphic in an open connected subset Ω of the complex plane and let γ be a positively oriented simple* *loop in* Ω *along which* h *is analytic. Then*

$$\frac{1}{2\pi i}\int_\gamma h(t)dt = \sum_s \mathrm{Res}[h(t); t = s],$$

where $\sum_s$ *denotes the sum over all poles* s *of* h *enclosed by* γ.

A variety of consequences can be derived form the residue theorem. Maybe the most important concerns coefficients of analytic functions (see e.g. [39, Theorem IV.4]):

Theorem 2.11 (Cauchy's coefficient formula)**.** *Let* g *be analytic in an open connected subset* Ω *of the complex plane containing* 0 *and let* γ *be a positively oriented simple loop around* 0 *in* Ω. *Then, the coefficient* $[t^n]g(t)$ *admits the integral representation*

$$a_n \equiv [t^n]g(t) = \frac{1}{2\pi i}\int_\gamma g(t)\frac{dt}{t^{n+1}}.$$

This formula allows to deduce information about the coefficients from the properties of the function itself, using suitably chosen contours of integration. It is thus possible to estimate the coefficients $[t^n]g(t)$ in the expansion of g near 0 by using information of g away from 0.

To investigate the correspondence between the asymptotic expansion of the coefficients of a function and the behavior of the function near its singularities, it is necessary to distinguish two cases: functions that have a moderate growth or decay at their dominant singularities in contrast to those that exhibit some form of exponential growth. In the first case, *singularity analysis* is the right method for studying the coefficients, while in the second case it is the so-called *saddle-point analysis.*

Singularity analysis

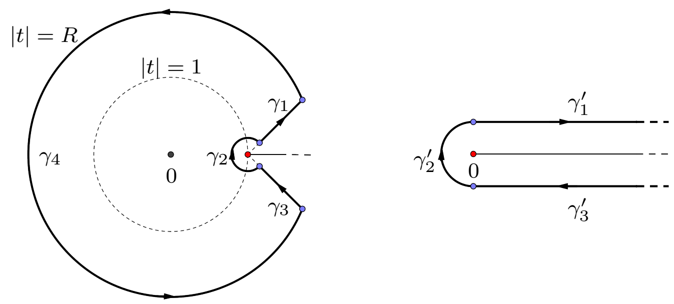
In Chapters 3 and 4 where we study several properties of the generalized Ewens measure we will apply *singularity analysis* to investigate the coefficients of generating functions of interest. Here, based on Section VI.3 in [39], we outline a general approach to the analysis of coefficients of generating functions whose singular expansions involve fractional powers and logarithms. Then in Sections 3.2 and 4.2, we present in detail how this method is adapted to our model.

The main ingredient of singularity analysis, a theory which was developed by Flajolet and Odlyzko in [38], is Cauchy's coefficient formula combined with special contours of integration known as *Hankel contours*: they come very close to the singularity and then move away; hereby, they capture the essential asymptotic information contained in the singularity.

Without loss of generality, let us assume in this section that the dominant singularity is located at $\rho = 1$. Indeed, if $g(t)$ has radius of convergence 1, then $g(t/\rho)$ has radius of convergence ρ and $[t^n]g(t/\rho) = \rho^{-n}[t^n]g(t)$. For a first intuition, consider $g(t) = (1-t)^{-\alpha}$ for some $\alpha \in \mathbb{R} \setminus \mathbb{Z}_{\leq 0}$. Cauchy's coefficient formula yields

$$[t^n](1-t)^{-\alpha} = \frac{1}{2\pi i}\int_\gamma (1-t)^{-\alpha}\frac{dt}{t^{n+1}}.$$

The basic strategy is as follows: choose a contour of integration γ as in Figure 1(a) with radius of the inner circle γ_2 being $1/n$.



(a) $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$ (b) $\gamma' = \gamma'_1 \cup \gamma'_2 \cup \gamma'_3$

Figure 1: The curves used for estimating the coefficients.

The integral along the large circle γ_4 with radius $R > 1$ decreases with a rate of order R^{-n}, as $n \to \infty$. Thus, we can let tend R to infinity and we are left with the curve $\gamma_1 \cup \gamma_2 \cup \gamma_3$. With the change of variables

$$t \mapsto 1 + w/n, \qquad dt \mapsto \frac{1}{n}dw \tag{2.9}$$

the curve γ is transformed into γ' in Figure 1(b) where γ'_1 and γ'_3 have distance 1 from the positive real axis. The integrand becomes

$$(1-t)^{-\alpha} \mapsto n^\alpha(-w)^{-\alpha}, \qquad \frac{1}{t^{n+1}} \mapsto e^{-w}\Big(1 + \mathcal{O}\Big(\frac{1}{n}\Big)\Big)$$

and altogether

$$[t^n](1-t)^{-\alpha} = \frac{n^{\alpha-1}}{2\pi i}\int_{\gamma'} e^{-w}(-w)^{-\alpha}dw\Big(1 + \mathcal{O}\Big(\frac{1}{n}\Big)\Big) = \frac{n^{\alpha-1}}{\Gamma(\alpha)}\Big(1 + \mathcal{O}\Big(\frac{1}{n}\Big)\Big).$$

The last equality is due to Hankel's formula for the Gamma function; see for example [39, Section B.3]. This strategy is easily extended to functions that involve logarithmic terms. That is, typically for functions of the form

$$g(t) = (1-t)^{-\alpha}\left(\frac{1}{t}\log\left(\frac{1}{1-t}\right)\right)^\beta \tag{2.10}$$

where $\alpha \in \mathbb{R} \setminus \mathbb{Z}_{\leq 0}$ and $\beta \in \mathbb{R}$, the coefficients satisfy the asymptotic

$$[t^n]g(t) = \frac{n^{\alpha-1}}{\Gamma(\alpha)}\log^\beta(n)\Big(1 + \mathcal{O}\Big(\frac{1}{\log(n)}\Big)\Big).$$

For a proof of this and the above statement; see Theorem VI.I and VI.2 in [39].

Now let us extend this method to situations where (2.10) only holds approximately. That is, we want to transfer an asymptotic expansion of a function near a singularity to an asymptotic expansion of its coefficients. For this purpose, we need to assume that the approximation is valid in a domain beyond the disc of convergence. Specifically, we introduce the Δ_0-domain.

Definition 2.12. *Let $R > 1$ and $0 < \phi < \frac{\pi}{2}$ be given. We define the set*

$$\Delta_0 \coloneqq \Delta_0(1, R, \phi) \coloneqq \{t \in \mathbb{C}; |t| < R, t \neq 1, |\arg(t-1)| > \phi\}.$$

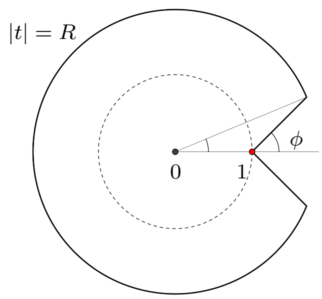


Figure 2: Illustration of Δ_0

The functions of interest are those which have an *algebraic-logarithmic* singularity at $t = 1$ and whose asymptotic expansion is valid in a Δ_0-domain.

Theorem 2.13 ([38],TheoremVI.3)**.** *Let α and β be arbitrary real numbers and let g be a function that satisfies the following two conditions:*

(1) g is holomorphic in $\Delta_0(1, R, \phi)$ for some $R > 1$ and $0 < \phi < \frac{\pi}{2}$,

(2) g satisfies in the intersection of a neighborhood of 1 with its $\Delta_0(1, R, \phi)$-domain the approximation

$$g(t) = \mathcal{O}\Big(\big(1-t\big)^{-\alpha} \log^{\beta}\Big(\frac{1}{1-t}\Big)\Big).$$

Then, one has $[t^n]g(t) = \mathcal{O}(n^{\alpha-1}\log^{\beta}(n))$.

For a detailed proof; see [39, SectionVI.3]. The basic idea is similar to what we discussed above. Let the radius of the outer circle γ_4 in Figure 1(a) be slightly smaller than R in $\Delta_0(1, R, \phi)$; then with condition (1), one can easily show that the contribution of the integral along γ_4 is negligible. Since the radius of the inner circle γ_2 is $1/n$, notice that the length of the contour γ_2 is $\mathcal{O}(1/n)$ and t^{-n-1} is $\mathcal{O}(1)$ on γ_2. Thus, the integral along γ_2 is of order $n^{\alpha-1}\log^{\beta}(n)$. The study of the rectilinear parts γ_1 and γ_2 is more complicated. Here condition (2) is needed as well as a change of variables as in (2.9).

This principle of transfer of error terms from functions to coefficients is valid in more general situations, for example for functions that have *finitely many* singularities on their circle of convergence. In this case, the contribution of each singularity is investigated by the basic singularity analysis process and the whole outcome is basically the sum of the individual contributions. We will apply this technique in Section 4.6.

A natural question to ask is how one can verify whether a function g satisfies conditions (1) and (2) in Theorem 2.13. Generally, many commonly encountered functions involving algebraic-logarithmic terms turn out to be Δ_0-analytic. It is usually not very difficult to identify the radius of convergence of a generating series, but it is not obvious to show the Δ_0-analyticity. One possibility is to apply Lindelöf's integral representation.

Theorem 2.14 (Lindelöf's integral representation)**.** *Let $\psi(z)$ be a holomorphic function for $z \in \mathbb{C}$ with* $\mathrm{Re}(z) > 0$ *which satisfies*

$$|\psi(z)| < Ce^{A|z|} \qquad \textit{for } |z| \to \infty,\ \mathrm{Re}(z) \geq \frac{1}{2}$$

for some constant $C > 0$ and $A \in (0, \pi)$. Then the radius of convergence of the function $g(t) := \sum_{k=1}^{\infty} \psi(k)(-t)^k$ is at least e^{-A} and

$$g(t) = -\frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} \psi(z) t^z \frac{\pi}{\sin(\pi z)} dz.$$

Furthermore, g can be holomorphically continued to the sector $|\arg(t)| < (\pi - A)$.

The proof of this Theorem is based on the residue theorem; see for example [37]. Theorem 2.14 is a useful tool to prove analyticity in a domain Δ_0, but it does not give any information about the asymptotic behavior of g near its singularity. The so-called Mellin transform is one way to get this information; see [29] for an introduction to this technique.

Saddle-point analysis

While singularity analysis is a method to investigate the asymptotic behavior of coefficients of functions having moderate growth, saddle-point analysis applies to functions which exhibit some form of *exponential growth* at their singularities. We will make use of this approach in Chapter 5, where we study the order of permutations with polynomial cycle weights $\theta_m = m^\gamma$, $\gamma > 0$. The starting point is again Cauchy's integral formula. Then, basically, the principle of the saddle-point method is to choose a path crossing a saddle-point, estimating the integrand locally near this saddle-point and deduce an asymptotic expansion of the integral itself. Since the saddle-point corresponds locally to a maximum of the integrand along the path, it is natural to expect that a small neighborhood of the saddle-point captures the relevant contribution of the integral. Some form of *concentration condition* will ensure this property.

Based on [39, Section VIII.5], we introduce the notion of *Hayman-admissibility*, which defines a wide class of functions to which saddle-point analysis is relevant. The interested reader is referred to [39, Section VIII] for the general theory.

Instead of generating series of the type (2.8), it is convenient to consider

$$G(t) \coloneqq \exp(g(t)) = \sum_{n=0}^{\infty} G_n t^n.$$

Assume that G is analytic at the origin. We want to estimate its coefficients by Cauchy's coefficient formula. A change to polar coordinates $t = e^{i\phi}$ yields

$$G_n \equiv [t^n]G(t) = \frac{1}{2\pi i}\int_\gamma G(t)\frac{dt}{t^{n+1}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(re^{i\phi})\frac{d\phi}{(re^{i\phi})^n}.$$

Let γ be a circle around the origin. We want to determine the radius such that γ crosses (or comes very close to) a saddle-point. Thus, we need to find the zero of the derivative of the integrand $F(t) \coloneqq G(t)\,t^{-n}$:

$$F'(t) = 0 \qquad \Longleftrightarrow \qquad tg'(t) = n. \tag{2.11}$$

We refer to this as the *saddle-point equation* and denote its solution by r_n. It grants us locally a quadratic approximation without linear terms. The saddle-point method is based on the following steps:

- Split the integration contour: $\gamma = \gamma_0 + \gamma_1$. The central part γ_0 is an arc of the circle which crosses the saddle point r_n (or passes very near to it) and is determined by $|\phi| \leq \phi_0$ for some suitably chosen ϕ_0.
- The integrand should be well approximated by a quadratic function along γ_0.
- The contribution of the remaining part γ_1 should be negligible.
- Under these conditions, the integral is asymptotically equivalent to an *incomplete* Gaussian integral. Introducing only negligible error terms, it should be asymptotically equivalent to a *complete* Gaussian integral, which is evaluable in a closed form.

The crucial point is to choose γ_0 large enough so that it captures the main contribution of the whole integral but small enough so that the integrand can be suitably reduced to its quadratic expansion. Let us make this concept more precise. The Taylor expansion of g around r yields

$$g(t) = g(r) + \sum_{m=1}^{\infty} a_m(r)\frac{(i\phi)^m}{m!}$$

where

$$\alpha(r) := a_1(r) = rg'(r) \qquad \text{and} \qquad \beta(r) := a_2(r) = r^2 g''(r) + rg'(r). \tag{2.12}$$

Thus, assuming that $g(r) + i\phi\alpha(r) - \frac{\phi^2}{2}\beta(r)$ is a good approximation for $g(t)$ on γ_0 and that the saddle-point equation (2.11) is satisfied, roughly, the asymptotics

$$\int_{-\pi}^{\pi} G(re^{i\phi})e^{-in\phi}d\phi \sim \int_{-\phi_0}^{\phi_0} G(r_n e^{i\phi})e^{-in\phi}d\phi \sim G(r_n)\int_{-\phi_0}^{\phi_0} e^{-\frac{\phi^2}{2}\beta(r_n)}d\phi$$

hold. Under the condition $\phi^2\beta(r_n) \to \infty$, this gives

$$\int_{-\pi}^{\pi} G(re^{i\phi})e^{-in\phi}d\phi \sim \frac{G(r_n)}{\sqrt{\beta(r_n)}}\int_{-\phi_0\sqrt{\beta(r_n)}}^{\phi_0\sqrt{\beta(r_n)}} e^{-\frac{s^2}{2}}ds \sim \frac{G(r_n)}{\sqrt{\beta(r_n)}}\int_{-\infty}^{\infty} e^{-\frac{s^2}{2}}ds = \frac{2\pi G(r_n)}{\sqrt{\beta(r_n)}}.$$

Altogether,

$$G_n \equiv [t^n]G(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(re^{i\phi})\frac{d\phi}{(re^{i\phi})^n} \sim \frac{G(r_n)}{r_n^n\sqrt{2\pi\beta(r_n)}}.$$

This is the concept of saddle-point analysis. Functions amenable to this technique are the so-called *Hayman-admissible* functions; see [39, Definition VIII.1].

Definition 2.15 (Hayman-admissibility)**.** *Let $G(t)$ have radius of convergence $\rho > 0$ and be always positive on some subinterval (R_0, ρ) of $(0, \rho)$. Then G is said to be Hayman-admissible if, with $\alpha(r)$ and $\beta(r)$ defined in* (2.12)*, it satisfies the following three conditions*

H1 [Capture condition] $\lim_{r\to\rho}\alpha(r) = +\infty$ *and* $\lim_{r\to\rho}\beta(r) = +\infty$.

H2 [Locality condition] *For some function $\phi_0(r)$ defined over (R_0, ρ) and satisfying $0 < \phi_0 < \pi$, one has*

$$G(re^{i\phi}) \sim G(r)e^{i\phi\alpha(r) - \phi^2\beta(r)/2} \qquad \text{as } r \to \rho,$$

uniformly in $|\phi| \le \phi_0(r)$.

H3 [Decay condition] *Uniformly in $\phi_0(r) \le |\phi| < \pi$*

$$G(re^{i\phi}) = o\left(\frac{G(r)}{\sqrt{\beta(r)}}\right).$$

It can easily be verified that the function $G(t =)e^t$ is Hayman-admissible with $\rho = +\infty$ and that $G(t) = e^{1/(1+t)}$ is Hayman-admissible with $\rho = 1$. In Chapter 5, we will present a slightly different version of admissibility, namely *log-admissibility*, where the conditions are imposed on $g(t)$ instead of on $G(t) = \exp(g(t))$. In fact, if $g(t)$ is log-admissible, then $G(t)$ is Hayman-admissible. We consider the function $g_\Theta(t) = \sum_{m=1}^{\infty}\frac{\theta_m}{m}t^m$ with $\theta_m = m^\gamma$, $\gamma > 0$, and show that it is log-admissible. In particular, for $\gamma = 1$, we get $g_\Theta(t) = 1/(1-t)$ which is indeed log-admissible.

Coefficients of Hayman-admissible functions can be systematically analyzed with the following theorem:

Theorem 2.16. *Let G be a Hayman-admissible function and let r_n be the unique solution of the saddle-point equation* (2.11)*. Then, as $n \to \infty$,*

$$G_n \equiv [t^n]G(t) \sim \frac{G(r_n)}{r_n^n \sqrt{2\pi\beta(r_n)}}.$$

The idea for the proof was sketched above; for a detailed proof see [39, Theorem VIII.4]. We will apply a slightly different version of this theorem in Chapter 5.

2.3 Mod-φ convergence

A new type of convergence of random variables was introduced in 2011 by Jacod et al. in [47]: *mod-Gaussian convergence*. It has interesting applications when typically the sequence of random variables X_n under consideration does not converge in distribution, meaning that the sequence of characteristic functions does not converge pointwise to a limit characteristic function, but nevertheless, the characteristic functions decay precisely like those of a suitable Gaussian G_n. Specifically, the convergence

$$\mathbb{E}[e^{itG_n}]^{-1}\mathbb{E}[e^{itX_n}] \to \Phi(t) \tag{2.13}$$

holds locally uniformly for $t \in \mathbb{R}$, where the limiting function Φ is continuous on $\mathbb{R}$ with $\Phi(0) = 1$. Therefore, the main idea is to find a natural renormalization of the characteristic functions of random variables rather than a renormalization of the random variables itself. Intuitively, X_n resembles a sum $G_n + Y_n$ where Y_n is a convergent sequence independent of G_n. However, in many interesting cases X_n does not exhibit this simple decomposition.

The convergence in (2.13) indeed appears in a variety of settings. Originally, it was inspired by issues of random matrix theory and number theory, namely by the connection between characteristic polynomials of large random matrices and the moments of the Riemann zeta function; see [47, 54]. The approach is based on probabilistic and harmonic analysis techniques and the field of applications ranges from the value distribution of the Riemann zeta function on the critical line (see [47, 54]) to L-functions over finite fields (see [54]).

It is natural to consider mod-φ convergence with respect to other suitable laws φ. In particular, it turns out that *mod-Poisson convergence* appears in analytic number theory in the context the classical Erdős-Kác theorem; recall (1.15). In [53] Kowalski and Nikeghbali discuss this connection in detail; notably, it is explained how mod-Poisson convergence takes into account the dependence structure of the framework under study. Moreover, Nikeghbali and Zeindler [60] prove mod-Poisson convergence for the total number of cycles of weighted random permutations.

More unexpectedly, *mod-Cauchy convergence* arises in [26] in one approach to the windings of a planar Brownian motion. Remarkably, all those examples of arithmetic, combinatorial and probabilistic nature are handled in a consistent way with relatively elementary tools. In a series of papers [26, 36, 47, 54, 53], properties and implications of mod-φ convergence were studied. Clearly, (2.13) implies a central limit theorem for X_n but it also yields other applications. Of particular interest for this thesis are a general local limit theorem presented by Delbaen et al. in [26], as well as a precise large deviations estimate explained in [36] by Féray et al.

In following paragraph we make a brief excursion into history to understand the mathematical setting in which mod-φ convergence appeared. This discussion has no immediate relation to the actual topic of this thesis; the reader may browse this paragraph, or skip directly to the second paragraph, where we present an overview of properties and applications of mod-φ convergence which will be relevant to establish our results.

The origin of mod-Gaussian convergence

Recall that the Riemann zeta function is defined for all complex s with $Re(s) > 1$ as a Dirichlet series or an Euler product:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \Big(1 - \frac{1}{p^s}\Big)^{-1}.$$

In 1859, Riemann published his famous conjecture which asserts that all the non-trivial zeros of the *Riemann zeta function* lie on the so-called *critical line*, that is they have real part 1/2; see [67]. This statement is one of the most famous open problems in mathematics and it is widely believed to be true. One of the greatest achievements providing an auspicious way to approach the *Riemann hypothesis* is due to Montgomery [58], who observed in conjunction with Dyson in 1973, loosely speaking, that the distribution of the eigenvalues of large random matrices is close to that of the non-trivial Riemann zeta zeros. This amazing phenomenon is perhaps the most striking discovery about the zeta function since Riemann. Since the late 1990's, the connection between random matrix theory and number theory has had renewed interest, mainly due to the work of Katz and Sarnak [48] on the one hand, and Keating and Snaith [49] on the other hand. Katz and Sarnak have proved the Montgomery conjecture for function fields of zeta functions, for which the Riemann conjecture in fact is proven. Keating and Snaith have pushed further the analogy between random matrix theory and number theory by providing a *moments conjecture* for the Riemann zeta function. The 2λ-th absolute moment of $\zeta(1/2 + it)$ is defined as

$$I_\lambda(T) := \frac{1}{T}\int_0^T |\zeta(1/2+it)|^{2\lambda} dt, \qquad \text{for } \lambda \in \mathbb{N}.$$

Since the early 20^{th} century mathematicians have tried to compute $\lim_{T\to\infty} I_\lambda(T)$. There is an important conjecture saying that $m(\lambda)$ defined by

$$\lim_{T\to\infty} \frac{1}{(\log(T))^{\lambda^2}} I_\lambda(T) = a(\lambda) m(\lambda) \tag{2.14}$$

exists, where $a(\lambda)$ is an Euler product, but the structure of $m(\lambda)$ is unknown. Obviously, $m(0) = 1$. In 1918 Hardy and Littlewood [44] proved that $m(1) = 1$, and in 1926, Ingham [46] proved that $m(2) = 1/12$. No other values are known. Based on number-theoretical arguments, it is believed that $m(3) = 42/9!$ and $m(4) = 24024/16!$; see [22] and [23]. It was shown by Goldston [40] in 1987 that the moments of $\log \zeta(1/2 + it)$ split asymptotically into two terms, one coming from random matrix theory and the other one coming from the primes. It is plausible to expect a similar behavior for the moments of $\zeta(1/2 + it)$. However, it remains a much-studied but still unsolved problem to show that $m(\lambda)$ is indeed a term coming from random matrix theory.

The factor $a(\lambda)$ is obtained by purely number theoretic considerations. Roughly speaking, it comes out of a probabilistic model of primes where the primes are thought of as behaving *independently* of each other. It is known that this model is wrong but it often captures the genuine behavior of arithmetic functions at the level of the central limit theorem. However, a correction term is needed which is given by $m(\lambda)$. But what is the source of this term?

Given its success in describing the statistical properties of the zeros of the Riemann zeta function (remember Montgomery's conjecture), it is natural to ask whether the value distribution of the zeta function may as well be investigated with the help of random unitary matrices. The question to be answered is: *which property of a random matrix plays the role of the zeta function?*

Since the zeta zeros are distributed like the eigenvalues of random unitary matrices, the zeta function might be expected to be similar, in terms of its value distribution, to the function whose zeros are the eigenvalues, that is, to the *characteristic polynomial* of such matrices. This idea was introduced and investigated by Keating and Snaith [49]. Consider $U(N)$ the group of random unitary matrices of size $N \times N$. Recall that U is a unitary matrix if $UU^* = I$ where U^* is the conjugate transpose of U and I is the identity matrix. For a matrix $A \in U(N)$ the characteristic polynomial $\mathcal{Z}$ is defined by

$$\mathcal{Z}(A, \theta) := \det(I_N - Ae^{-i\theta}),$$

where I_N denotes the identity matrix of size $N \times N$. It turned out that, concerning the value distribution, a complete characterization was aimed for $\log \zeta(1/2 + it)$. There is for example a beautiful theorem due to Selberg, which states that for any rectangle $B \subset \mathbb{C}$,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \mathbb{1}\left\{ \frac{\log \zeta(1/2 + it)}{\sqrt{\log\log(T)/2}} \in B \right\} dt = \frac{1}{2\pi} \iint\limits_B e^{-\frac{x^2+y^2}{2}} dx dy. \tag{2.15}$$

In other terms, if our probability space is the interval $[0, 1]$ equipped with the Lebesgue measure du, then the random variable $\log \zeta(1/2 + iuT)/\sqrt{\log\log(T)/2}$ converges in distribution, as $T \to \infty$, to a Gaussian distribution on $\mathbb{C}$, centered with unit variance.

Concerning the random matrix model, Keating and Snaith translated this result in terms of characteristic polynomials as follows. For any rectangle $B \subset \mathbb{C}$,

$$\lim_{N\to\infty} \int_{U(N)} \mathbb{1}\left\{ \frac{\log \mathcal{Z}(\mathcal{A},\theta)}{\sqrt{\log(N)/2}} \in B \right\} d_{\mu_{U(N)}}(A) = \frac{1}{2\pi} \iint\limits_{B} e^{-\frac{x^2+y^2}{2}} dxdy. \tag{2.16}$$

This theorem corresponds exactly to Selberg's theorem (2.15); notice that the scaling in (2.15) and (2.16) coincide if we set $N = \log(T)$. It is thus natural to assume that random matrix theory, in the limit as $N \to \infty$, can indeed model the value distribution of $\log(\zeta(1/2 + it))$ as $T \to \infty$.

At this point, the question arises whether the characteristic polynomial of large random unitary matrices can also be applied to solve the mystery about $m(\lambda)$ in (2.14). In terms of characteristic polynomials and by taking again $N = \log(T)$, the question which is analogous to (2.14) is whether

$$m_U(\lambda) := \lim_{N\to\infty} \frac{1}{N^{\lambda^2}} \int_{U(N)} |\mathcal{Z}(A,\theta)|^{2\lambda} dA \tag{2.17}$$

exists and, and if it does, which values it takes. Keating and Snaith [49] proved that for any complex number λ with $Re(\lambda) > -1$, m_U does indeed exist and that it takes the form

$$m_U(\lambda) = \frac{(G(1+\lambda))^2}{G(1+2\lambda)}.$$

Here, G denotes the Barnes G-function, and by its basic properties one can deduce

$$m_U(n) = \prod_{j=0}^{n-1} \frac{j!}{(j+n)!}.$$

Thus, one can compute $m_U(0) = 1$, $m_U(1) = 1$, $m_U(2) = 1/12$, $m_U(3) = 42/9!$ and $m_U(4) = 24024/16!$. Hence, $m_U(k) = m(k)$ for $k = 1, 2$ and conjecturally for $k = 3, 4$. Then, they make the following conjecture for the moments of the Riemann zeta function: for any complex number λ with $Re(\lambda) > -1$, one should have $m(\lambda) = m_U(\lambda)$, namely

$$\lim_{T\to\infty} \frac{1}{(\log(T))^{\lambda^2}} \frac{1}{T} \int_0^T |\zeta(1/2+it)|^{2\lambda} dt = a(\lambda) \frac{(G(1+\lambda))^2}{G(1+2\lambda)}. \tag{2.18}$$

Here, $a(\lambda)$ is, as already mentioned, an *arithmetic factor* defined by an Euler product and $m(\lambda) = \frac{(G(1+\lambda))^2}{G(1+2\lambda)}$ is the so-called *random matrix factor*. This extra factor should be considered as a correction term to take into account the fact that the prime numbers do *not* behave independently of each other.

Conjecture (2.18) is supported by numerical data [62]. It has created a new philosophy, called the *Keating-Snaith philosophy*, to produce conjectures in number theory by computing analogous quantities in the random matrix theory world where calculations are often more tractable.

Recently, Kowalski and Nikeghbali, together with several coauthors, (see [12, 26, 47, 54, 53]) have proposed a new approach to understand the role of random matrix theory in these conjectures, inspired by looking at expression (2.17) in a slightly different way. Let X_N be uniformly distributed on $U(N)$ and Y_N be distributed like the characteristic polynomial of X_N at zero, that is like $\det(I_N - X_N)$. Notice that (2.17) can be rephrased as follows:

$$\lim_{N\to\infty} \frac{1}{N^{\lambda^2}} \mathbb{E}\left[|Y_N^{2\lambda}|\right] = \frac{(G(1+\lambda))^2}{G(1+2\lambda)}.$$

Now they substituted $\lambda = iu$ and $Z_N = \log(|Y_N|^2)$ and obtained

$$\lim_{N\to\infty} e^{u^2 \log(N)} \mathbb{E}\left[e^{iuZ_N}\right] = \frac{(G(1+iu))^2}{G(1+2iu)}.$$

Thus, by (2.13), the sequence $(Z_N)_{N\in\mathbb{N}}$ is *mod-Gaussian convergent* with parameters $(0, 2\log(N))$ and limiting function

$$\Phi(u) = \frac{(G(1+iu))^2}{G(1+2iu)}.$$

In this framework, again by taking $\lambda = iu$, the moments conjecture (2.18) can be rephrased as follows: the random variable $\log|\zeta(1/2+iU_T)|^2$, where U_T is a variable uniformly distributed on $(0,T)$, converges as $T \to \infty$ in the mod-Gaussian sense with parameters $(0, 2\log\log(T))$ and limiting function $a(\lambda)m(\lambda)$. In other terms,

$$\lim_{T\to\infty} e^{u^2 \log\log(T)} \mathbb{E}\left[e^{iu\log|\zeta(1/2+iU_T)|^2}\right] = a(\lambda)m(\lambda). \tag{2.19}$$

In fact it turns out that the situation where the limiting function can be split into a product of two factors, one coming from some group and the other from a naive probabilistic model involving primes, is not specific to the Riemann zeta function. It also appears in the study of the arithmetic function $\omega(n)$ which counts the number of distinct prime divisors of an integer n and the celebrated *Erdős-Kác theorem* (1.15): it states that $\omega(n)$ behaves for large n like a Gaussian random variable with mean $\log\log(n)$ and variance $\log\log(n)$. This phenomenon where increasing variance is observed suggests, in the context of the mod-Gaussian approach, to investigate the behavior of $\omega(n)$ without normalizing. Indeed, it turns out that $\omega(n)$, suitable modified, converges in mod-Poisson sense with a limiting function which decomposes, analogously to (2.19), into two factors, one being an Euler product which one would expect to be the entire limit function if the primes would behave perfectly independent. However, to compensate the dependence structure of the prime model, a second factor appears.

Motivated by these examples, the concept of mod-φ convergence was developed. The established theorems look like higher order central limit theorems: they take into account the dependence structure of the framework under study.

Properties of mod-φ convergence

Let us now discuss the definitions and basic properties of mod-φ convergence. We present briefly the main features which will be of interest for this thesis and refer the reader to [47, 53] and the references mentioned below for more details.

Definition 2.17 ([47], Definition 1.1)**.** *The sequence X_n is said to converge in the mod-Gaussian sense if the convergence*

$$\lim_{n\to\infty} e^{-it\mu_n+t^2\sigma_n^2/2}\mathbb{E}[e^{itX_n}] = \Phi(t)$$

holds for all $t \in \mathbb{R}$, where $\mu_n \in \mathbb{R}$ and $\sigma_n^2 \geq 0$ are two sequences and Φ is a complex-valued function which is continuous at 0 (notice that necessarily $\Phi(0) = 1$). We call (μ_n, σ_n^2) the parameters and Φ the limiting function.

Intuitively, this definition suggests that X_n is close in some sense to a Gaussian random variable. In most applications this notion is of interest for $\sigma_n \to \infty$. If the mod-Gaussian convergence is sufficiently uniform, then, to some extend, X_n is indeed normal distributed. To specify this feature, let us denote by $\mathrm{d_K}(X, Y)$ the Kolmogorov distance of two real-valued random variables X and Y:

$$\mathrm{d_K}(X, Y) := \sup_{x\in\mathbb{R}} \big|\mathbb{P}[X \leq x] - \mathbb{P}[Y \leq x]\big|. \tag{2.20}$$

The following asymptotic holds (see [54, Remark 3]):

Proposition 2.18. *Let the sequence X_n be mod-Gaussian convergent with parameters $(0, \sigma_n^2)$ such that $\sigma_n^2 \to \infty$ as $n \to \infty$. Assume further that the limiting function is of $\mathcal{C}^1$-class and that the convergence holds in $\mathcal{C}^1$-topology. Then*

$$\mathrm{d_K}(X_n, G_n) = \mathcal{O}(\sigma_n^{-1})$$

holds, where G_n is a centered Gaussian random variable with variance σ_n^2.

In a variety of examples where a central limit theorem is observed with increasing variance (such as for the total cycle number in (1.9), the order of permutations in (1.11) or the Erős-Kác theorem in (1.15)) the framework of mod-Gaussian convergence suggests to study the involved random variables X_n without normalizing. However, whenever X_n is *integer-valued* with variance $\mathbb{V}[X_n] \to \infty$, then it does not converge in the mod-Gaussian sense; see [47, Proposition 4.11]. This is due to the fact that $\mathbb{E}[e^{itX_n}]$ is 2π-periodic for all $n \in \mathbb{N}$. Nevertheless, it often turns out that in these situations, the random variables under consideration are mod-Poisson convergent.

Definition 2.19 ([53], Definition 2.1)**.** *The sequence X_n is said to converge in the mod-Poisson sense with parameters λ_n if the limit*

$$\lim_{n\to\infty} e^{\lambda_n(1-e^{it})}\mathbb{E}[e^{itX_n}] = \Phi(t)$$

exists for all $t \in \mathbb{R}$, and the convergence is locally uniform. The limiting function Φ is then continuous at 0 and $\Phi(0) = 1$.

Notice that mod-Poisson convergence with growing parameters implies mod-Gaussian convergence. More precisely, assume that the sequence X_n is mod-Poisson convergent with parameters $\lambda_n \to \infty$. Then the sequence

$$\frac{X_n - \lambda_n}{\lambda_n^{1/3}} \tag{2.21}$$

is mod-Gaussian convergent with parameters $(0, \lambda_n^{1/3})$ and limit function $\Phi(t) = e^{t^3/6}$. However, when comparing the renormalized sequence to a Gaussian law, the Poisson nature of the sequence X_n gets lost. Specifically, similar to Proposition 2.18, assuming that the mod-Poisson convergence is sufficiently uniform one can show that X_n is indeed close to a Poisson random variable; see [53, Proposition 2.5] or [12, Proposition 3.1].

Apart from comparing the probability $\mathbb{P}[X_n \leq x]$ of a mod-Gaussian or mod-Poisson sequence X_n with the respective probability of the reference law, it is of interest to study $\mathbb{P}[X_n \in B]$ for compact sets B when the sequence X_n is mod-φ convergent, where the reference law is a fairly general probability distribution. It is known in classical probability theory that, under suitable assumptions, a central limit theorem for independent and identical distributed random variables $(X_n)_{n\geq 1}$ implies a local limit theorem, meaning that asymptotics for $\mathbb{P}[X_1 + \cdots + X_n \in B]$ are available as $n \to \infty$ where B is a Jordan measurable set. Delbaen et al. present in [26] a general framework where mod-φ convergence in $\mathbb{R}^d$, under suitable conditions, implies a local limit theorem in $\mathbb{R}^d$, for arbitrary dimension d. We focus on the one-dimensional setting; in Section 4.3 we will establish a local limit theorem for the order of weighted random permutations based on the main result in [26], which we now present.

Consider a probability measure μ on $\mathbb{R}$ with characteristic function φ and a sequence of real-valued random variables X_n with characteristic functions φ_n. To establish a local limit theorem for general mod-φ convergence, the following conditions are needed:

H1 The characteristic function φ of the measure μ is integrable; in particular, μ has a continuous density $d\mu/dm$ with respect to the Lebesgue measure m.

H2 There exist a sequence of real numbers $(\beta_n)_{n\geq 1}$ such that $\beta_n \to \infty$ and the renormalized sequence X_n/β_n converges in distribution to μ.

H3 For all $k \geq 0$, the sequence

$$f_{n,k}(t) := \varphi_n(t\beta_n^{-1})\, \mathbb{1}_{|t\beta_n^{-1}|\leq k}$$

is uniformly integrable on $\mathbb{R}$.

Notice that condition **H1** excludes discrete probability laws, such as Poisson distributions. Condition **H3** is necessary in the proof of Theorem 2.21 below to pass at a certain point from pointwise convergence to convergence in L^1.

Definition 2.20 ([26], Definition 1). *If μ is a probability measure on $\mathbb{R}$ with characteristic function φ, X_n a sequence of real random variables with characteristic functions φ_n and if the conditions* **H1**, **H2** *and* **H3** *hold, then the sequence X_n is called mod-φ convergent.*

The main result in [26] shows that, when X_n is mod-φ convergent, the behavior of $\mathbb{E}[f(X_n)]$ for reasonable functions f is well-controlled:

Theorem 2.21 ([26], Theorem 5). *Suppose that mod-φ convergence holds for the sequence X_n. Then, as $n \to \infty$,*

$$\sigma_n \mathbb{E}[f(X_n)] \to \frac{d\mu}{dm}(0) \int_{\mathbb{R}} f(x)dx$$

holds for all continuous functions with compact support. Consequently, as $n \to \infty$,

$$\sigma_n \mathbb{P}[X_n \in B] \to \frac{d\mu}{dm}(0)\, m(B)$$

holds for relatively compact Borel sets $B \subset \mathbb{R}$ with $m(\partial B) = 0$ (that is for bounded Jordan measurable sets $B \subset \mathbb{R}$).

A variety of applications of this local limit theorem are presented in [26, Section 3]; they range from the winding number of a planar Brownian motion and the characteristic polynomial of random matrices to the density of the values of the Riemann zeta function on the critical line. In Section 4.3 of this thesis we present a further application to the order of weighted random permutations.

At the end of this section we present the link of mod-φ convergence and precise large deviations established by Féray et al. in [36]. The framework is as follows: assume that X_n is a sequence of random variables such that the corresponding moment generating functions $\varphi_n(t) = \mathbb{E}[e^{tX_n}]$ exist in a strip $S_c := \{t \,|\, -c < \mathrm{Re}(t) < c\}$ where c is a positive real number. Assume further that there exists an infinitely divisible distribution with moment generating function $\varphi(t) = \exp(\eta(t))$ and an analytic function Φ which does not vanish on the real part of S_c, such that the convergence

$$\lim_{n\to\infty} \exp(-\beta_n \eta(t))\varphi_n(t) = \Phi(t)$$

holds locally uniformly for $t \in S_c$ and let β_n be some sequence tending to infinity. In [36] the authors provide large deviations estimates for the case of lattice and the non-lattice random variables. For our study, we will only need the following theorem.

Theorem 2.22 ([36],Theorem 3.2). *Suppose that φ is non-lattice and that X_n is a sequence of non-lattice random variables which satisfies the assumptions above. Denote by h the solution of $\eta'(h(x)) = x$ and let F be the Fenchel-Legendre transform, that is $F(x) = hx - \eta(h)$. If x is in the range of $\eta'_{|(0,c)}$, then*

$$\mathbb{P}[X_n \geq \beta_n x] = \frac{\exp(-\beta_n F(x))}{h\sqrt{2\pi\beta_n \eta''(h)}} \Phi(h)\big(1 + o(1)\big).$$

2.4 Number theoretic sums

We recall the asymptotic behavior of some averages over multiplicative functions involving the von Mangoldt function Λ, which will be particularly useful to study the properties of the order of a permutation; see Chapter 4 and 5. The von Mangoldt function Λ is defined as

$$\Lambda(n) = \begin{cases} \log(p) & \text{if } n = p^k \text{ for some prime } p \text{ and } k \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

The Chebyshev function ψ is given by

$$\psi(x) := \sum_{k \leq x} \Lambda(k) = \sum_{p^k \leq x} \log(p). \tag{2.22}$$

By definition, the prime number theorem is equivalent to

$$\psi(x) = x\big(1 + o(1)\big) \qquad \text{as } x \to \infty. \tag{2.23}$$

A more precise explicit formula which was proved by Mangoldt is given by

$$\psi(x) = x - \sum_{\varrho} \frac{x^{\varrho}}{\varrho} - \log(2\pi) - \frac{1}{2}\log(1 - x^{-2}), \tag{2.24}$$

where the sum is taken over the non-trivial zeros of the Riemann zeta function; see [72, Section II.4.3]. Then the Riemann hypothesis is equivalent to

$$\psi(x) = x + \mathcal{O}(x^{1/2+\epsilon}) \qquad \text{for all } \epsilon > 0; \tag{2.25}$$

see [72, Section II.4, Corollary 3.1]. The relation of $\psi(n)$ and the least common multiple of the numbers $1, 2, ..., n$ is given by

$$\operatorname{lcm}(1, 2, ..., n) = \exp(\psi(n)).$$

Furthermore, by [2, Theorem 4.9], as $x \to \infty$,

$$\sum_{k \leq x} \frac{\Lambda(k)}{k} = \log(x) + \mathcal{O}(1), \tag{2.26}$$

and by (2.23) this can be generalized for $0 \neq \alpha \neq 1$ to

$$\begin{aligned} \sum_{k=x}^{y} \Lambda(k) k^{-\alpha} &= \sum_{k=x}^{y} \Lambda(k) \int_k^y \alpha t^{-\alpha-1} dt + y^{-\alpha} \sum_{k=x}^{y} \Lambda(k) \\ &= \alpha \int_x^y \sum_{k=x}^{t} \Lambda(k) t^{-\alpha-1} dt + y^{-\alpha}(y - x + o(y)) \\ &= \frac{1+\alpha}{1-\alpha}\big(y^{1-\alpha} - x^{1-\alpha}\big)(1 + o(1)). \end{aligned} \tag{2.27}$$

Let us finish up this section with a summary of asymptotics of functions which we encounter frequently in this thesis. The upper incomplete gamma function is defined as

$$\Gamma(a,y) := \int_y^\infty x^{a-1}\, e^{-x}\, dx$$

and satisfies

$$\Gamma(a,y) = \Gamma(a) - \frac{1}{a}y^a + \Sigma_2(a,y) \qquad \text{as } y \to 0, \tag{2.28}$$

with

$$\Sigma_j(a,y) = \sum_{k=j}^{\infty}(-1)^k \frac{y^{k-1+a}}{(k-1)!(k-1+a)}. \tag{2.29}$$

On the other hand,

$$\Gamma(a,y) = e^{-y}y^{a-1}(1+\mathcal{O}(1/y)) \qquad \text{as } y \to \infty. \tag{2.30}$$

Furthermore, the Error function is defined as

$$\operatorname{erf}(x) := \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$$

and satisfies

$$\operatorname{erf}(x) = 1 + \mathcal{O}(x^{-1}e^{-x^2}) \qquad \text{as } x \to \infty \tag{2.31}$$

and

$$\operatorname{erf}(x) = -1 + \mathcal{O}(x^{-1}e^{-x^2}) \qquad \text{as } x \to -\infty. \tag{2.32}$$

Finally, recall also the Euler-Maclaurin formula

$$\sum_{m=1}^{b} f(m) = \int_1^b f(x)dx + \int_1^b (x - \lfloor x \rfloor) f'(x)dx + f(b)(b - \lfloor b \rfloor). \tag{2.33}$$

For example, it yields

$$\sum_{m=1}^{b} \frac{\log(m)}{m} = \frac{\log^2(b)}{2} + \mathcal{O}\Big(\frac{\log(b)}{b}\Big). \tag{2.34}$$

3

A_n-permutations under the generalized Ewens measure

This chapter is devoted to the so-called *generalized Ewens parameters* and presents results that were obtained jointly with Nikeghbali and Zeindler in [59]. Roughly speaking, this measure comprises all types of parameters such that the generating series $g_\Theta := \sum_{m=1}^{\infty} \frac{\theta_m}{m} t^m$ exhibits logarithmic singularities. We will apply singularity analysis to generalize results on uniform permutations to this model.

The objects of interest are the so-called A-permutations, which are permutations that can be decomposed into cycles whose lengths are all in a set A. They have been studied for more than thirty years with respect to the uniform and Ewens measure; see [74] for a long list of references. We extend some classical results, such as a limit theorem for the large cycles and a functional central limit theorem for the total cycle number, to the weighted measure $\mathbb{P}_\Theta$. In particular, $\mathbb{P}_\Theta$ allows us to consider A_n-permutations, where the sets A_n depend on the degree n of the symmetric group $\mathfrak{S}_n$.

In Section 3.4 we show that the size ordered cycle lengths converge in law to a Poisson-Dirichlet distribution. This agrees with the results by Kingman [50] and Shmidt and Vershik [68], who studied the same asymptotic behavior with respect to the Ewens measure. Furthermore, in Section 3.5 we consider the number of cycles in a permutation with lengths not exceeding n^x for $0 \leq x \leq 1$ and show that this process converges, after normalization, to a standard Brownian motion. This extends results of Delaurentis and Pittel [25] and Hansen [43] to the A_n-weighted measure.

We apply tools from combinatorics and complex analysis. An advantage of our method is that it is very flexible and it allows us to study quantities of interest under further restrictions, for example the total number of the even cycles; see Section 3.6.

3.1 Preliminaries

A-permutations are classical objects in combinatorics. It is well-known that with respect to the uniform measure and for a wide class of sets A, the behavior of A-permutations is similar to those of the whole permutation group. To give two simple examples, recall the definition of the cycle counts C_m in (1.1) and the total number of cycles T_n in (1.8):

$$C_m := \#\{i : \lambda_i = m\} \quad \text{and} \quad T_n := \sum_{m=1}^{n} C_m. \tag{3.1}$$

In [75] Yakymiv proved that for a wide class of sets A with positive density, as $n \to \infty$ the cycle counts C_m converge in distribution for $m \in A$ to independent Poisson distributed random variables Z_m with expectation $1/m$; this behavior coincides with (1.4). For the same model was proved that the total number of cycles whose length belongs to A satisfies a central limit theorem similar to (1.9) but with the density of A appearing as a factor in the renormalization; see [76].

In all previous studies on A-permutations one has only investigated its behavior under the uniform or the Ewens measure and with the set A being independent of the degree n of the permutation. Here, we consider a more general A_n-weighted measure. Recall from Section 2.1 that we denote by $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_\ell)$ the *cycle-type* of σ and by $\ell = \ell(\lambda)$ its *length*.

Definition 3.1. *Let $A_n \subset \{1, \ldots, n\}$ and $\Theta = (\theta_m)_{m \geq 1}$ be given, with $\theta_m \geq 0$ for every $m \geq 1$. We define the A_n-weighted measure of $\sigma \in \mathfrak{S}_n$ as*

$$\mathbb{P}_\Theta^{(A_n)}[\sigma] := \frac{1}{h_n n!} \prod_{m=1}^{\ell(\lambda)} \theta_{\lambda_m} \mathbb{1}_{\{\lambda_m \in A_n\}} = \frac{1}{h_n n!} \prod_{m=1}^{n} \theta_m^{C_m} \mathbb{1}_{\{m \in A_n\}} \tag{3.2}$$

with $h_n = h_n(A_n)$ a partition function that makes $\mathbb{P}_\Theta^{(A_n)}$ a probability distribution; set $h_0 := 1$.

Notice that for $A_n = \mathbb{N}$ one obtains the weighted measure $\mathbb{P}_\Theta$ as defined in Definition 1.1. Define furthermore

$$D_n := \{1, \ldots, n\} \setminus A_n \quad \text{and} \quad d_n := \begin{cases} \max D_n & \text{if } D_n \neq \emptyset, \\ 1 & \text{otherwise}. \end{cases} \tag{3.3}$$

We study the behavior of the measure $\mathbb{P}_\Theta^{(A_n)}$ for $d_n = o(n)$, that is the cycle lengths not contained in A_n grow slowly (the precise assumptions on d_n can be found in Theorem 3.6 below). This assumption is motivated by a model in [53, Section 6] about mod-Poisson convergence for an analog of the Erdős-Kác theorem (see (1.15)) for polynomials over finite fields.

The asymptotic behavior of random variables on the group $\mathfrak{S}_n$ with respect to the weighted measure $\mathbb{P}_\Theta^{(A_n)}$ strongly depends on the sequence $\Theta = (\theta_m)_{m \geq 1}$. For

$A_n = \mathbb{N}$, the link of the coefficients $(h_n)_{n\geq 1}$ and the generating series g_Θ is given in Corollary 2.4:

$$\sum_{n=0}^{\infty} h_n t^n = \exp(g_\Theta(t)) \quad \text{where} \quad g_\Theta(t) = \sum_{m=1}^{\infty} \frac{\theta_m}{m} t^m. \tag{3.4}$$

For general $A \subset \mathbb{N}$ we need the following lemma:

Lemma 3.2. *Let $A \subset \mathbb{N}$ and $\Theta = (\theta_m)_{m\geq 1}$ be given as in Definition 3.1 and define $D := \mathbb{N} \setminus A$. We then have as formal power series*

$$\sum_{n=0}^{\infty} h_n(A) t^n = \exp\left(g_\Theta(t) - L_D(t)\right), \tag{3.5}$$

where $g_\Theta(t)$ is as in (3.4) and $L_D(t)$ is the formal power series

$$L_D(t) := \sum_{m\in D} \frac{\theta_m}{m} t^m.$$

Proof. Combine the definition of h_n, Lemma 2.1 and Lemma 2.3 to get

$$h_n(A) = \frac{1}{n!} \sum_{\sigma\in\mathfrak{S}_n} \prod_{m=1}^{n} \left(\theta_m \mathbb{1}_{\{m\in A\}}\right) = \sum_{\lambda\vdash n} \frac{1}{z_\lambda} \prod_{m=1}^{n} \left(\theta_m \mathbb{1}_{\{m\in A\}}\right).$$

Consequently,

$$\sum_{n=1}^{\infty} h_n(A) t^n = \exp\left(\sum_{m=1}^{\infty} \frac{\theta_m}{m} \mathbb{1}_{\{m\in A\}}\, t^m\right) = \exp\left(g_\Theta(t) - L_D(t)\right).$$

□

Thus, with the notation of Definition 2.2, we have

$$h_n(A) = [t^n]\Big[\exp\big(g_\Theta(t) - L_D(t)\big)\Big].$$

Recall the method of singularity analysis illustrated in Section 2.2. We will choose the parameters $\Theta = (\theta_m)_{m\geq 1}$ such that we can apply this method in order to study the behavior of $h_n(A)$ as $n \to \infty$. The Δ_0-domain which was introduced in Definition 2.12 for generating functions having radius of convergence 1 is considered here for general radii of convergence.

Definition 3.3. *Let $0 < \rho < R$ and $0 < \phi < \frac{\pi}{2}$ be given. We define the set*

$$\Delta_0 := \Delta_0(\rho, R, \phi) := \{t \in \mathbb{C}; |t| < R, t \neq \rho, |\arg(t-\rho)| > \phi\}.$$

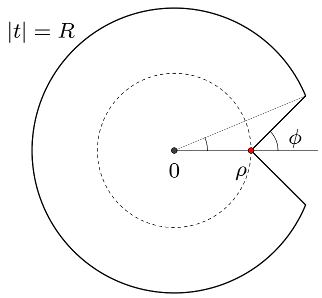


Figure 3: Illustration of Δ_0

Let us now introduce the *generalized Ewens measure*. Rather than defining conditions for the parameters $\Theta = (\theta_m)_{m \geq 1}$ directly, we will impose them on the generating function g_Θ. In view of conditions (1) and (2) in Theorem 2.13, we require that g_Θ is analytic a the Δ_0-domain and that it admits logarithmic growth at its dominant singularity.

Definition 3.4. *Let $\rho, \vartheta > 0$ and $K \in \mathbb{R}$ be given. We write $\mathcal{F}(\rho, \vartheta, K)$ for the class of all functions g satisfying the two conditions*

(1) g is holomorphic in $\Delta_0(\rho, R, \phi)$ for some $R > \rho$ and $0 < \phi < \frac{\pi}{2}$,

(2)

$$g(t) = \vartheta \log \left(\frac{1}{1 - t/\rho} \right) + K + \mathcal{O}(t - \rho) \qquad \text{as } t \to \rho \text{ for } \ t \in \Delta_0. \tag{3.6}$$

Notice that constant parameters $\theta_m = \vartheta$ lead to $g_\Theta(t) = -\vartheta \log(1-t) \in \mathcal{F}(1, \vartheta, 0)$ and thus the Ewens measure is covered by the family $\mathcal{F}(\rho, \vartheta, K)$. More generally, functions of the form $g(t) = -\vartheta \log(1-t) + f(t)$ with f holomorphic for $|t| < 1 + \epsilon$ are contained in the class $\mathcal{F}(1, \vartheta, f(1))$. In particular, the case $\theta_m \neq \vartheta$ for only finitely many m is included in $\mathcal{F}(1, \vartheta, \cdot)$.

Remark 3.5. The justification for the name *generalized Ewens measure* relies on the following observation. Theorem VI.4 in [39] implies that if g_Θ is defined as in (3.4) and the parameters θ_m are such that g_Θ belongs to $\mathcal{F}(\rho, \vartheta, K)$, then there exists some ϵ_m such that

$$\theta_m \rho^m = \vartheta + \epsilon_m \quad \text{ with } \ \epsilon_m \to 0 \ \text{ and } \ \sum_{m=1}^{\infty} \frac{|\epsilon_m|}{m} < \infty. \tag{3.7}$$

For $A_n = \mathbb{N}$, this class of parameters has been recently studied by several authors. The asymptotic Ewens case $\theta_m \to \vartheta$ (which corresponds to the class $\mathcal{F}(1, \vartheta, K)$) was studied for example by Ercolani and Ueltschi [30]. They prove that the length of a typical cycle has order n and that the expected total number of cycles is asymptotically equal to $\vartheta \log(n)$; see [30, Theorem 6.1].

For the general class $\mathcal{F}(\rho, \vartheta, K)$, Nikeghbali and Zeindler [60] apply singularity analysis to obtain the asymptotic behavior of h_n and estimates for the characteristic function of the total number of cycles T_n, among others. For T_n, they generalize the central limit theorem (1.17) to the generalized Ewens measure, that is they prove

$$\frac{T_n - \vartheta \log(n)}{\sqrt{\vartheta \log(n)}} \xrightarrow{d} \mathcal{N}(0,1)$$

as $n \to \infty$, for all parameters such that g_Θ belongs to $\mathcal{F}(\rho, \vartheta, K)$; see [60, Theorem 4.2]. In fact, they are able to prove a finer convergence for T_n, namely mod-Poisson convergence; see Definition 2.19. From there, they apply results on mod-Poisson convergence to T_n to obtain Poisson approximation (see [60, Lemma 4.6]) and large deviations estimates (see [60, Theorem 4.7]).

These works are complemented by results we obtained jointly with Nikeghbali and Zeindler in [59] and which are presented in the subsequent sections. We apply a similar singularity analysis method as in [60]. However, our approach is more general since we allow restrictions on the cycle length.

3.2 Singularity analysis for increasing cycle lengths

The aim of this section is to provide by means of complex analysis arguments a tool, Theorem 3.6, that allows us to compute the asymptotic behavior as $n \to \infty$ of $h_n(A_n)$ and of other quantities of interest considered with respect to the generalized A_n-weighted Ewens measure.

Theorem 3.6. *Let $g(t)$ belong to $\mathcal{F}(\rho, \vartheta, K)$ and $(D_n^{(j)})_{n\in\mathbb{N}, 1\le j\le k}$ with $D_n^{(j)} \subset \{1, \ldots, n\}$ be given. We define*

$$d_n^{(j)} := \begin{cases} \max D_n^{(j)} & \text{if } D_n^{(j)} \neq \emptyset, \\ 1 & \text{otherwise}\,, \end{cases} \qquad \text{and} \qquad \bar{d}_n := \max\{d_n^{(j)}\}.$$

Let further

$$G_n(t, w, v_1, \ldots, v_k) := \exp\left(wg(t) + \sum_{j=1}^{k} v_j L_{D_n^{(j)}}(t) \right) \tag{3.8}$$

with $w, v_1, \ldots, v_k \in \mathbb{C}$ and $L_{D_n^{(j)}}(t)$ as in Lemma 3.2. Suppose that for each $C \in \mathbb{R}$

$$C \log(n) - \frac{n}{d_n} \to -\infty \tag{3.9}$$

holds as $n \to \infty$. Then we have for any fixed $b \in \mathbb{N}$

$$\begin{aligned} &[t^{n-b}]\,[G_n(t, w, v_1, \ldots, v_k)] \\ &\qquad = \frac{e^{Kw} n^{w\vartheta - 1}}{\rho^{n-b}} \exp\left(\sum_{j=1}^{k} v_j L_{D_n^{(j)}}(\rho) \right) \left(\frac{1}{\Gamma(w\vartheta)} + \mathcal{O}\left(\frac{\bar{d}_n}{n} \right) \right) \end{aligned}$$

uniformly for bounded $|w|, |v_1|, \dots, |v_k| \le r$ *for some* $r > 0$.

Remark 3.7.
- We have introduced in Theorem 3.6 a family of sets $(D_n^{(j)})$ since this will allow us to compute easily the finite dimensional distributions of the process B_n in Section 3.5. However, in most cases, we will have $j = 1$ and then we denote $D_n := D_n^{(1)}$, $d_n := d_n^{(1)} = \overline{d}_n$ and $v := v_1$.
- Notice that (3.9) is satisfied for example for $d_n \sim \log(n)$ or $d_n \sim n^\alpha$ with $0 < \alpha < 1$. Furthermore, assumption (3.9) is not satisfied if $D_n^{(j)} = \{1, \dots, n\}$ for some j. However, in this case

$$[t^n]\left[\exp\left(vL_{D_n^{(j)}}(t)\right)\right] = [t^n]\left[\exp\left(vg(t)\right)\right]$$

  holds and we can thus handle $D_n^{(j)} = \{1, \dots, n\}$ with Theorem 3.6 by replacing $L_{D_n^{(j)}}(t)$ with $g(t)$ in (3.8).
- In most cases we will apply Theorem 3.6 with $b = 0$, except in Section 3.4, where we study the behavior of the large cycles.

Before proving Theorem 3.6, we deduce the asymptotic behavior of h_n as $n \to \infty$.

Corollary 3.8. *Let* $g_\Theta(t)$ *be as in* (3.4) *and assume that it belongs to* $\mathcal{F}(\rho, \vartheta, K)$. *Let* $(A_n)_{n\in\mathbb{N}}$ *be the defining sets of the measures* $\mathbb{P}_\Theta^{(A_n)}$ *in Definition 3.1 and let* D_n *and* d_n *be as in* (3.3). *If the sequence* d_n *satisfies the assumption* (3.9), *then*

$$h_n(A_n) = \exp\left(-L_{D_n}(\rho)\right)\frac{n^{\vartheta-1}e^K}{\rho^n\Gamma(\vartheta)}\left(1 + \mathcal{O}\left(\frac{d_n}{n}\right)\right).$$

Proof. We know from Lemma 3.2 that for arbitrary sets $A \subset \mathbb{N}$

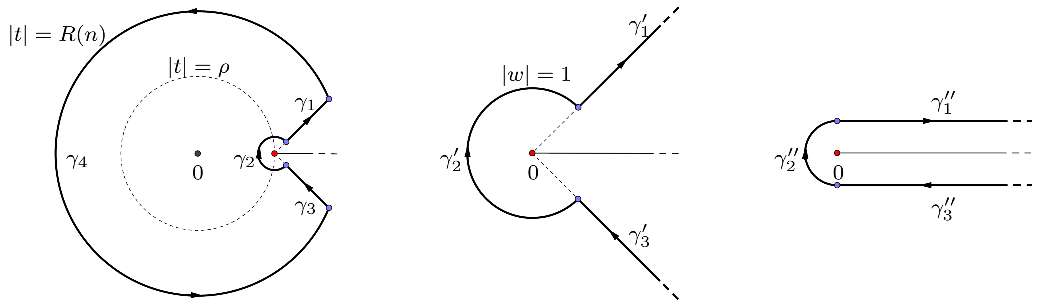
$$h_n(A) = [t^n]\left[\exp\left(g_\Theta(t) - L_D(t)\right)\right]$$

holds. Thus the corollary follows immediately from Theorem 3.6. □

Proof of Theorem 3.6. Let us assume $k = 1, b = 0$ and write $D_n := D_n^{(1)}, d_n := \bar{d}_n = d_n^{(1)}$ and $v := v_1$; the proof of the general case is very similar. We apply Cauchy's integral formula (see Theorem 2.11) to $G_n(t, w, v)$. This gives

$$[t^n]\left[G_n(t, w, v)\right] = \frac{1}{2\pi i}\int_\gamma \exp\left(wg(t) + vL_{D_n}(t)\right)\frac{dt}{t^{n+1}} \tag{3.10}$$

for some curve γ. We follow the idea in [39, Section VI.3] (which was also sketched in Section 2.2) and choose the curve γ as in Figure 4(a). The main difference with [39] is that we let the radius of the large circle slowly tend to ρ while it is fixed in [39]. More precisely, by assumption $g(t)$ is holomorphic in $\Delta_0(\rho, R, \phi)$ (see Definition 3.3)



(a) $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$ (b) $\gamma' = \gamma_1' \cup \gamma_2' \cup \gamma_3'$ (c) $\gamma'' = \gamma_1'' \cup \gamma_2'' \cup \gamma_3''$

Figure 4: The curves used in the proof of Theorem 3.6.

and continuous on $\overline{\Delta_0(\rho, R, \phi)} \setminus \{\rho\}$. We then define the radius of the large circle as

$$R(n) \coloneqq \min\{\rho(1 + 1/d_n), R\}$$

and define further the curve γ as follows:

$$\begin{aligned}
\gamma_1(x) &\coloneqq \rho\,(1 + xe^{i\phi}) && \text{for } \ x \in [1/n, \hat{r}_n],\\
\gamma_2(\varphi) &\coloneqq \rho\,\big(1 - e^{-i\varphi}/n\big) && \text{for } \ \varphi \in [-\pi + \phi, \pi - \phi],\\
\gamma_3(x) &\coloneqq \rho\,(1 + \big(\hat{r}_n - x\big)e^{-i\phi}) && \text{for } \ x \in [0, \hat{r}_n - 1/n],\\
\gamma_4(\varphi) &\coloneqq R(n)e^{i\varphi} && \text{for } \ \varphi \in [-\pi + \alpha_n, \pi - \alpha_n],
\end{aligned}$$

where α_n and $\hat{r}_n$ are chosen such that the curve γ is closed, i.e. $\rho + \hat{r}_n e^{i\phi} = R(n)e^{i\alpha_n}$.

Let us first compute the integral along the outer circle γ_4. If $\sup_n d_n = C < \infty$, we clearly have $R(n) \geq \widetilde{R} > \rho$ for some $\widetilde{R}$ independent of n. Thus all points of the curve γ_4 have at least a distance $|\widetilde{R} - \rho| > 0$ from ρ. Therefore $g(t)$ is uniformly bounded on γ_1. Furthermore $L_{D_n}(t)$ involves only θ_m with $m \leq C$ and is thus also uniformly bounded. Consequently,

$$\left|\frac{1}{2\pi i}\int_{\gamma_1} \exp\left(wg(t) + vL_{D_n}(t)\right)\,\frac{dt}{t^{n+1}}\right| = \mathcal{O}(\widetilde{R}^{-n}) = \mathcal{O}\left(\frac{n^{w\vartheta-2}}{\rho^n}e^{vL_{D_n}(\rho)}\right).$$

If $\sup_n d_n = \infty$ we have to be more careful. In this case

$$\begin{aligned}
&\left|\frac{1}{2\pi i}\int_{\gamma_1} \exp\left(wg(t) + vL_{D_n}(t)\right)\,\frac{dt}{t^{n+1}}\right|\\
&\leq \frac{1}{2\pi R(n)^n}\int_{-\pi+\alpha_n}^{\pi-\alpha_n} \left|\exp\left(wg\big(R(n)e^{i\varphi}\big) + vL_{D_n}\big(R(n)e^{i\varphi}\big)\right)\right|\,d\varphi\\
&\leq \frac{1}{R(n)^n}\exp\big(\max_{-\pi+\alpha_n\leq\varphi\leq\pi-\alpha_n}\{\mathrm{Re}\left[wg\big(R(n)e^{i\varphi}\big) + vL_{D_n}\big(R(n)e^{i\varphi}\big)\right]\}\big).
\end{aligned}$$

Using that $g(t)$ belongs to $\mathcal{F}(\rho, \vartheta, K)$ and hence is continuous on $\overline{\Delta_0(\rho, R, \phi)} \setminus \{\rho\}$ and the expansion of $g(t)$ around ρ, one immediately obtains

$$\mathrm{Re}(g(t)) \leq \vartheta\log\left|\frac{1}{1 - t/\rho}\right| + \mathcal{O}(1) \ \text{ and } \ \mathrm{Im}(g(t)) = \mathcal{O}(1) \quad \text{for all } t \in \Delta_0 \setminus \{\rho\}\,.$$

This yields

$$\operatorname{Re}[wg\big(R(n)e^{i\varphi}\big)] \le |\operatorname{Re}(w)|\vartheta \log\left|\frac{1}{1-R(n)/\rho}\right| + \mathcal{O}(1) = |\operatorname{Re}(w)|\vartheta\log(d_n) + \mathcal{O}(1).$$

Furthermore, we get

$$\begin{aligned}\left|L_{D_n}\left((1+1/d_n)\,\rho e^{i\varphi}\right)\right| &\le \sum_{m\in D_n}\frac{\theta_m}{m}\rho^m\left(1+1/d_n\right)^m\\ &\le \sum_{m\in D_n}\frac{\theta_m}{m}\rho^m\left(1+\mathcal{O}\left(m/d_n\right)\right)\\ &\le L_{D_n}(\rho)+\mathcal{O}(1)\le \log(d_n)+\mathcal{O}(1)\end{aligned} \tag{3.11}$$

since $m \le d_n$ and $\theta_m\rho^m \sim \vartheta$; see (3.7). We also have

$$R(n)^{-n} \le \rho^{-n}(1+1/d_n)^{-n} = \rho^{-n}\exp(-n\log(1+1/d_n)) \le \rho^{-n}\exp\left(-\frac{n}{2d_n}\right).$$

Combining the above computations, we obtain

$$\left|\frac{1}{2\pi i}\int_{\gamma_1} e^{wg(t)+vL_{D_n}(t)}\,\frac{dt}{t^{n+1}}\right| \quad = \mathcal{O}\left(\rho^{-n}\exp\left(-\frac{n}{2d_n}+(|\operatorname{Re}(w)|+|v|)\vartheta\log(d_n)\right)\right).$$

It remains to prove that this is $\mathcal{O}(n^{w\vartheta-2}\exp\big(-vL_{D_n}(\rho)\big)$. This holds if

$$-\frac{n}{2d_n}+(|\operatorname{Re}(w)|\vartheta+|v|)\log(d_n) \le \operatorname{Re}((w\vartheta-2)\log(n)-vL_{D_n}(\rho))+\mathcal{O}(1)$$

but this follows immediately from assumption (3.9) since

$$L_{D_n}(\rho) \le \log(d_n) \le \log(n).$$

The computations for the integrals along γ_1, γ_2 and γ_3 are completely similar to the computations in the proof of Theorem VI.3 in [39] (see also a sketch of the proof in Section 2.2) and we thus give only a short overview. A simple calculation gives

$$L_{D_n}\big(\gamma_2(\varphi)\big) = L_{D_n}(\rho)+\mathcal{O}\left(d_n/n\right) \quad \text{and} \quad L_{D_n}\big(\gamma_3(x)\big) = L_{D_n}(\rho)+\mathcal{O}\left(d_n x\right). \tag{3.12}$$

These observations together with the computations in [39] imply

$$\frac{1}{2\pi i}\int_{\gamma_2\cup\gamma_3\cup\gamma_4} G_n(t,w,v)\,\frac{dt}{t^{n+1}} = \frac{1}{2\pi i}\frac{n^{w\vartheta-1}}{\rho^n}\int_{\gamma'} z^{-w\vartheta}e^z dt\ \ (1+\mathcal{O}\left(d_n/n\right))$$

with γ' as in Figure 4(b). The variable substitution $x=-z$ combined with a simple contour argument give

$$\frac{1}{2\pi i}\int_{\gamma'} z^{-w\vartheta}e^z dt = \frac{1}{2\pi i}\int_{\gamma''}(-x)^{-w\vartheta}e^{-x}dt = \frac{1}{\Gamma(w\vartheta)}$$

with γ'' as in Figure 4(c). We have used in the second equality that that the integral is a well know expression for the inverse of Γ-function. Further details can be found for instance in [39, Section B.3]. □

To investigate the behavior of the large cycles in Section 3.4 and a functional central limit theorem in Section 3.5 we have to consider expressions of the form

$$[t^n]\left[f(t)\cdot\exp\left(g(t)+vL_{D_n}(t)\right)\right],$$

where the function $f(t)$ is either a polynomial depending on n or it is independent of n and behaves like a derivative of the logarithm near ρ. By suitable modifications of Theorem 3.6 we obtain in these cases the following asymptotics.

Corollary 3.9. *Let the assumptions of Theorem 3.6 be fulfilled with $k=1$ and write $D_n := D_n^{(1)}, d_n := \bar{d}_n = d_n^{(1)}$. If $f(t)$ is holomorphic in a Δ_0-domain and there exists a constant $\beta \geq 0$ such that*

$$f(t) = (1-t/\rho)^{-\beta}(1+\mathcal{O}(t-\rho)), \qquad \text{as } t\to\rho \ \text{ for } \ t\in\Delta_0,$$

then

$$[t^n]\left[f(t)\cdot\exp\big(g(t)+vL_{D_n}(t)\big)\right] = \frac{e^K n^{\vartheta+\beta-1}}{\rho^n}\exp\big(vL_{D_n}(\rho)\big)\left(\frac{1}{\Gamma(\vartheta+\beta)}+\mathcal{O}\left(\frac{d_n}{n}\right)\right).$$

Proof. Since $g(t)$ belongs to $\mathcal{F}(\rho,\vartheta,K)$, the asymptotic (3.6) together with the assumtion of the corollary yield

$$\log f(t)+g(t) = -(\vartheta+\beta)\log(1-t/\rho)+K+\mathcal{O}\left(t-\rho\right) \qquad \text{as } t\to\rho.$$

Thus, $\log f(t)+g(t)$ belongs to $\mathcal{F}\big(\rho,\vartheta+\beta,K\big)$ and the corollary follows immediately from Theorem 3.6 with $g(t)$ replaced by $\log f(t)+g(t)$. □

Corollary 3.10. *Let the assumptions of Theorem 3.6 be fulfilled with $k=1$ and write $D_n := D_n^{(1)}, d_n := \bar{d}_n = d_n^{(1)}$. Let further $P_n(t)$ be a sequence of polynomials with*

$$P_n(t) = \sum_k p_{k,n}t^k$$

such that $P_n\big(\rho(1+1/d_n)\big) = P_n(\rho)(1+o(1))$. We then have for each $v\in\mathbb{R}$

$$[t^n]\left[P_n(t)\cdot\exp\left(g(t)+vL_{D_n}(t)\right)\right] = P_n(\rho)\frac{n^{\vartheta-1}}{\rho^n}\exp\big(vL_{D_n}(\rho)\big)\Big(\frac{1}{\Gamma(w\vartheta)}+o(1)\Big).$$

Proof. The proof is a simple variant of that of Theorem 3.6. Thus, we only illustrate the estimate over γ_4 with $\gamma_4(\varphi)=\rho(1+1/d_n)e^{i\varphi}$. Analogously to the first part of the proof of Theorem 3.6 we obtain

$$\begin{aligned}&\left|\frac{1}{2\pi i}\int_{\gamma_1}P_n(t)\exp\big(g(t)+vL_{D_n}(t)\big)\,\frac{dt}{t^{n+1}}\right|\\&\leq\frac{P_n\big(\rho(1+1/d_n)\big)}{2\pi R(n)^n}\int_{-\pi+\alpha_n}^{\pi-\alpha_n}\left|\exp\big(wg\big(R(n)e^{i\varphi}\big)+vL_{D_n}\big(R(n)e^{i\varphi}\big)\big)\right|\,d\varphi\end{aligned}$$

The latter integral is now the same as in the proof of Theorem 3.6. Using the estimate in the proof of Theorem 3.6 and the assumption on $P_n(t)$ then completes the proof. □

3.3 The cycle counts and the total number of cycles

It is natural to begin our study with the cycle counts C_m and the total number of cycles T_n as defined in (3.1). First, we compute their generating functions and then deduce with Theorem 3.6 the asymptotic behavior of the coefficients. As mentioned in the introduction, the required computations are quite similar to those in [60]. Therefore, we give here only a short overview and refer to [60] for more details.

Lemma 3.11. *Let $A \subset \mathbb{N}$, $D := \mathbb{N} \setminus A$ be given. We then have for $w \in \mathbb{C}$ as formal power series*

$$\sum_{n=1}^{\infty} h_n(A)\mathbb{E}_{\Theta}^{(A)}\left[\exp(wT_n)\right] t^n = \exp\left(e^w g_{\Theta}(t) - e^w L_D(t)\right).$$

Let further $M = \{m_1, \dots, m_d\} \subset A$ be given. We then have for $w_{m_1}, \dots, w_{m_d} \in \mathbb{C}$ as formal power series

$$\sum_{n=1}^{\infty} h_n(A)\mathbb{E}_{\Theta}^{(A)}\left[\exp\left(\sum_{j=1}^{d} w_{m_j} C_{m_j}\right)\right] t^n = \exp\left(\sum_{j=1}^{d} \frac{\theta_{m_j}}{m_j}(e^{w_{m_j}} - 1)t^{m_j}\right) e^{g_{\Theta}(t) - L_D(t)}.$$

The proof is a simple application of Lemma 2.3 and the computations are similar to those in the proof of Lemma 3.2. It follows with Lemma 3.11 that

$$h_n(A)\mathbb{E}_{\Theta}^{(A)}\left[\exp(wT_n)\right] = [t^n]\left[\exp\left(e^w g_{\Theta}(t) - e^w L_D(t)\right)\right]$$

with $A \subset \mathbb{N}$ arbitrary. We can thus replace A in the previous equation by any A_n depending on n; notice that this is not possible in Lemma 3.11. Now combine Theorem 3.6 and Lemma 3.11 to obtain the asymptotic behavior of the cycle counts.

Theorem 3.12. *Suppose that $g_{\Theta}(t)$ belongs to $\mathcal{F}(\rho, \vartheta, K)$. Let further $M = \{m_1, \dots, m_d\}$ and $(A_n)_{n \in \mathbb{N}}$ with $A_n \subset \{1, \dots, n\}$ be given and let d_n be defined as in* (3.3)*. Suppose that*

(1) d_n satisfies assumption (3.9) *and*

(2) there exists $n_0 \in \mathbb{N}$ such that $M \subset A_n$ for all $n \geq n_0$.

Then

$$\mathbb{E}_{\Theta}^{(A_n)}\left[\exp\left(\sum_{j=1}^{d} w_{m_j} C_{m_j}\right)\right] = \exp\left(\sum_{j=1}^{d} \frac{\theta_{m_j}}{m_j}(e^{w_{m_j}} - 1)\rho^{m_j}\right) + \mathcal{O}\left(\frac{\bar{d}_n}{n}\right) \quad (3.13)$$

uniformly in $w_{m_1}, \dots, w_{m_d}$ for bounded $\mathrm{Re}(w_{m_1}), \cdots, \mathrm{Re}(w_{m_d})$*. In particular, the random variables $C_{m_j}, m_j \in M$ converge in law to independent Poisson distributed random variables Z_{m_j} with $\mathbb{E}Z_{m_j} = \theta_{m_j}\rho^{m_j}/m_j$.*

Proof. Equation (3.13) follows immediately from Lemma 3.11 and Theorem 3.6. The error is uniform for bounded $\mathrm{Re}(w_{m_1}), \cdots, \mathrm{Re}(w_{m_d})$ since all $C_{m_j} \in \mathbb{N}$ and thus the function on the left-hand side of (3.13) is periodic. $\square$

The asymptotic behavior of the total cycle number T_n is computed analogously.

Theorem 3.13. *Let $g_\Theta(t)$, $(A_n)_{n\in\mathbb{N}}$ and d_n be defined as in Theorem 3.12. Then*

$$\mathbb{E}_\Theta^{(A_n)}\left[\exp(isT_n)\right] = n^{\vartheta(e^{is}-1)} e^{(K-L_{D_n}(\rho))(e^{is}-1)} \left(\frac{\Gamma(\vartheta)}{\Gamma(e^{is}\vartheta)} + \mathcal{O}\left(\frac{d_n}{n}\right)\right) \tag{3.14}$$

uniformly in s for bounded $\mathrm{Re}(is)$.

We will prove a more general result in Section 3.5; see Theorem 3.17. Given the characteristic function of the total cycle number, one can show the following central limit theorem, in analogy to Theorem 4.2 in [60].

Corollary 3.14. *Under the same assumptions of Theorem 3.12, we have as $n \to \infty$*

$$\frac{T_n - \vartheta \log(n)}{\sqrt{\vartheta \log(n)}} \xrightarrow{d} \mathcal{N}(0,1).$$

We will state and prove in Section 3.5 a more general result; see Corollary 3.21. In fact, still in analogy to [60], it follows immediately from equation (3.14) that T_n converges in a stronger sense, namely it is mod-Poisson convergent (see Definition 2.19).

Corollary 3.15. *Under the same assumptions as in Theorem 3.12, the sequence $(T_n)_{n\in\mathbb{N}}$ converges in the mod-Poisson sense with parameter $K + \vartheta\log(n) - L_{D_n}(\rho)$ and limiting function $\Gamma(\vartheta)/\Gamma(\vartheta e^{is})$.*

As in [60], one may now approximate T_n by a Poisson random variable with mean $K + \vartheta\log(n) - L_{D_n}(t)$ or compute large deviations estimates with Theorem 2.22. In Section 3.5 we state and prove an analogous result; see Corollary 3.19.

3.4 Behavior of large cycles

In this section we study the asymptotic behavior of the large cycles. The main result, Theorem 3.16, yields the same asymptotic behavior as in the Ewens case; see for instance Shmidt and Vershik [68] and Kingman [50]. For $\sigma \in \mathfrak{S}_n$, denote by $\ell^{(1)}(\sigma)$ the length of its longest cycle, by $\ell^{(2)}(\sigma)$ the length of its second longest cycle and so on. If σ has cycle type $\lambda = (\lambda_1, \lambda_2, \cdots)$, then $\ell^{(j)} = \lambda_j$ for $j \in \mathbb{N}$.

Theorem 3.16. *Let $(A_n)_{n\in\mathbb{N}}$ be the defining sets of the measures $\mathbb{P}_\Theta^{(A_n)}$ and let D_n and d_n be as in* (3.3)*. Suppose that g_Θ belongs to $\mathcal{F}(\rho, \vartheta, K)$, that d_n satisfies assumption* (3.9) *and that for all $b \geq 0$*

$$\left(\frac{\partial}{\partial t}\right)^{b+1} g_\Theta(t) = \frac{\vartheta b!}{\rho^{b+1}(1-t/\rho)^{b+1}}(1 + \mathcal{O}(t-\rho)) \tag{3.15}$$

holds as $t \to \rho$. We then have, as $n \to \infty$,

$$\left(\frac{\ell^{(1)}}{n}, \frac{\ell^{(2)}}{n}, \dots\right) \xrightarrow{d} \mathcal{PD}(\vartheta)$$

where $\mathcal{PD}(\vartheta)$ denotes the Poisson-Dirichlet distribution with parameter ϑ (see [20]).

Proof. Let $\ell_1 = \ell_1(\sigma)$ be the length of the cycle containing 1, $\ell_2 = \ell_2(\sigma)$ containing the least element not contained in the cycle containing 1 and so on. We prove that for each fixed $m \in \mathbb{N}$, as $n \to \infty$,

$$\left(\frac{\ell_1}{n}, \frac{\ell_2}{n-\ell_1}, \dots, \frac{\ell_m}{n - \sum_{j=1}^{m-1} \ell_j}\right) \xrightarrow{d} (B_1, \dots, B_m) \tag{3.16}$$

holds, where $B_1, \dots, B_m$ are independent beta random variables with parameter $(1, \vartheta)$, that is they have the density $(1-x)^{\vartheta-1}/\beta(1,\vartheta)$, where β is the beta function. This result immediately implies the assertion of the theorem; see for instance [68].

We start with the case $m = 1$. Let us first compute the distribution of ℓ_1. If $k \in A_n$ is given, then there are $(n-1)\cdots(n-k+1)$ possible cycles of length k containing the element 1, and the choice of such a cycle does not influence the cycle lengths of the remaining cycles. The definition of $h_n(A_n)$ together with a simple computation gives

$$\mathbb{P}_\Theta^{(A_n)}\left[\ell_1 = k\right] = \frac{\theta_k}{n} \frac{h_{n-k}(A_n)}{h_n(A_n)} \mathbb{1}_{\{k \in A_n\}}.$$

We use the Pochhammer symbol $(k)_b = k(k-1)\cdots(k-b+1)$ and get for $b \geq 1$

$$\mathbb{E}_\Theta^{(A_n)}\left[(\ell_1 - 1)_b\right] = \frac{1}{n} \sum_{k=b+1}^{n} (k-1)_b\, \theta_k\, \mathbb{1}_{\{k \in A_n\}} \frac{h_{n-k}(A_n)}{h_n(A_n)}.$$

On the other hand we have

$$\left(\frac{\partial}{\partial t}\right)^{b+1} g_\Theta(t) = \sum_{k=b+1}^{\infty} (k-1)_b\, \theta_k t^{k-b-1}.$$

This together with the definition of $L_{D_n}(t)$ and Lemma 3.2 yields

$$h_n(A_n)\mathbb{E}_\Theta^{(A_n)}\left[(\ell_1 - 1)_b\right] = [t^{n-b-1}]\left[\frac{e^{g_\Theta(t) - L_{D_n}(t)}}{n} \left(\frac{\partial}{\partial t}\right)^{b+1} \Big(g_\Theta(t) - L_{D_n}(t)\Big)\right].$$

Let us now apply Corollaries 3.9 and 3.10 to compute the asymptotic behavior of this expression. If follows with Corollary 3.9 and assumption (3.15) that

$$\begin{aligned} &[t^{n-b-1}]\left[\exp\left(g_\Theta(t) - L_{D_n}(t)\right) \left(\frac{\partial}{\partial t}\right)^{b+1} g_\Theta(t)\right] \\ &\qquad = \vartheta b! \frac{n^{\vartheta+b} e^K}{\rho^n \Gamma(\vartheta + b + 1)} \exp\left(-L_{D_n}(\rho)\right)\left(1 + \mathcal{O}(d_n/n)\right). \end{aligned} \tag{3.17}$$

We show next that the remaining part can be neglected with respect to (3.17). We get with (3.7)

$$L_{D_n}(\rho) = \sum_{k=1}^{d_n} \frac{\theta_k}{k}\rho^k = \mathcal{O}\left(\sum_{k=1}^{d_n} \frac{1}{k}\right) = \mathcal{O}(\log(d_n))$$

and

$$\left(\frac{\partial}{\partial t}\right)^{b+1} L_{D_n}(\rho) = \sum_{k=1}^{d_n}(k-1)_b\theta_k\rho^{k-b-1} = \mathcal{O}\left(\sum_{k=1}^{d_n}\binom{k}{b}\right) = \mathcal{O}\left(\binom{d_n}{b+1}\right) = \mathcal{O}\Big(d_n^{b+1}\Big).$$

Then, Corollary 3.10 together with this computations gives

$$[t^{n-b-1}]\left[e^{g_\Theta(t)-L_{D_n}(t)}\left(\frac{\partial}{\partial t}\right)^{b+1} L_{D_n}(t)\right] = \mathcal{O}\left(\frac{n^{\vartheta-1}(d_n)^{b+1}\exp\left(-L_{D_n}(\rho)\right)}{\rho^n}\right).$$

Comparing this error term to (3.17), we see that it is negligible since $d_n = o(n)$. Consequently, the leading term of $h_n(A_n)\mathbb{E}_\Theta^{(A_n)}\left[(\ell_1-1)_b\right]$ comes from (3.17) and combined with the asymptotic behavior of h_n (see Corollary 3.8) we obtain

$$\mathbb{E}_\Theta^{(A_n)}\left[(\ell_1-1)_b\right] = \vartheta n^b \frac{b!\,\Gamma(\vartheta)}{\Gamma(\vartheta+b+1)}\left(1+\mathcal{O}\left(\frac{d_n}{n}\right)\right).$$

It follows that

$$\mathbb{E}_\Theta^{(A_n)}\left[\left(\frac{\ell_1}{n}\right)^b\right] = \frac{b!\,\Gamma(\vartheta+1)}{\Gamma(\vartheta+b+1)} = \mathbb{E}[B_1^b]$$

with B_1 a beta random variable with parameter $(1,\vartheta)$. This completes the proof in the case $m=1$. Equation (3.16) now can be proved for arbitrary m by induction over m. The argument is (almost) the same as in the proof of Proposition 5.2 in [16]. One only has to check that

$$\mathbb{P}_{\Theta,n}^{(A_n)}\left[\frac{\ell_{m+1}}{n-\sum_{j=1}^m \ell_j} \le a_{m+1}\middle|\ell_1 = a_1,\ldots,\ell_m = a_m\right] = \mathbb{P}_{\Theta,\tilde{n}}^{(A_n)}\left[\frac{\ell_{m+1}}{\tilde{n}} \le a_{m+1}\right]$$

holds, where $\tilde{n} = n - \sum_{j=1}^m a_j$. □

3.5 A functional central limit theorem - without restriction

The object of this section is to prove that the number of cycles with length not exceeding n^x for $0 \le x \le 1$ converges, after normalization, weakly to the standard Brownian motion with respect to the Skorohod topology. (Details on the Skorohod topology and weak convergence of processes can be found for instance in [20]). Formally, this means we consider the functional

$$B_n(x) := \sum_{m=1}^{\lfloor n^x \rfloor} C_m. \tag{3.18}$$

It was first shown by DeLaurentis and Pittel [25], with respect to the uniform measure on $\mathfrak{S}_n$, that the process

$$W_n(x) := \frac{B_n(x) - x\log(n)}{\sqrt{\log(n)}} \tag{3.19}$$

converges, as $n \to \infty$, weakly to the standard Brownian motion for $0 \leq x \leq 1$. A corresponding result for the Ewens measure ($\theta_m = \vartheta$ for all $m \geq 1$) was shown by Hansen [43] and Donelly et al. [28]. For this, $\log(n)$ in (3.19) needs to be replaced by $\vartheta \log(n)$. By an appropriate rescaling, we will show in this section the validity of an analogous result for our more general measure $\mathbb{P}_\Theta^{(A_n)}$ with the usual assumptions on the generating function g_Θ.

Throughout this section, we assume no restrictions on the cycle lengths, that is we take $A_n = \{1, \ldots, n\}$ in Definition 3.1, and we write $\mathbb{E}_\Theta$ instead of $\mathbb{E}_\Theta^{(A_n)}$, $\mathbb{P}_\Theta$ instead of $\mathbb{P}_\Theta^{(A_n)}$ and h_n instead of $h_n(A_n)$. First, let us compute the characteristic function of the process given in (3.18).

Theorem 3.17. *Suppose that $g_\Theta(t)$ belongs to $\mathcal{F}(\rho, \vartheta, K)$ and let the process B_n be defined as in* (3.18)*. Then, for any fixed $0 \leq x < 1$, we have*

$$\mathbb{E}_\Theta\left[\exp(isB_n(x))\right] = \exp\big((e^{is} - 1)L_{D_x}(\rho)\big)\big(1 + \mathcal{O}(n^{x-1})\big), \tag{3.20}$$

with $D_x = \{1, \ldots, \lfloor n^x \rfloor\}$.

Remark 3.18. Notice that $B_n(1) = T_n$, and thus Theorem 3.13 states a similar behavior as in (3.20) for $x = 1$, except that the 1 on the right-hand side in (3.20) is replaced by the quotient $\Gamma(\vartheta)/\Gamma(e^{is}\vartheta)$.

Proof. Consider $B_b := \sum_{m=1}^{b} C_m$. With Lemma 2.1 we get

$$h_n \mathbb{E}_\Theta\left[\exp(isB_b)\right] = \sum_{\lambda \vdash n} \frac{1}{z_\lambda} \exp\left(\sum_{m=1}^{b} C_m\right) \prod_{m=1}^{\ell(\lambda)} \theta_{\lambda_m}.$$

Now apply Lemma 2.3 to obtain

$$\begin{aligned}\sum_{n=0}^{\infty} h_n \mathbb{E}_\Theta\left[\exp(isB_b)\right] t^n &= \sum_{\lambda} \frac{1}{z_\lambda}\left(\prod_{m=1}^{b} \left(\theta_m e^{is}\right)^{C_m}\right)\left(\prod_{m=b+1}^{\infty} \theta_m^{C_m}\right) t^{|\lambda|} \\ &= \exp\left(\sum_{m=1}^{b} \frac{\theta_m e^{is}}{m} t^m + \sum_{m=b+1}^{\infty} \frac{\theta_m}{m} t^m\right) \\ &= \exp\big((e^{is} - 1)L_{D_b}(t) + g_\Theta(t)\big),\end{aligned}$$

where $D_b = \{1, \ldots, b\}$. Then set $b = \lfloor n^x \rfloor$ and Theorem 3.6 gives the result. $\square$

Corollary 3.19. *Let g_Θ and B_n be as in Theorem 3.17. Then, for any fixed $0 \leq x < 1$, the sequence $(B_n(x))_{n\in\mathbb{N}}$ is strongly mod-Poisson convergent with limiting function 1 and parameter $L_{D_x}(\rho)$.*

Recall the definition of mod-Poisson convergence in Section 2.3. This corollary follows immediately from (3.20). Note again that a similar result for $x = 1$ can be found in Corollary 3.15. We obtain from (3.7) together with Euler's summation formula (2.33)

$$L_{D_x}(\rho) = \sum_{m=1}^{\lfloor n^x \rfloor} \frac{\vartheta}{m} + \sum_{m=1}^{\lfloor n^x \rfloor} \frac{\epsilon_m}{m} = x\vartheta \log(n) + c + o(1) \tag{3.21}$$

as $n \to \infty$ with some $c \in \mathbb{R}$. This shows that the mod-Poisson convergence in Corollary 3.20 does also hold with parameter $x\vartheta\log(n) + c$. Given this, we can estimate the distance of $B_n(x)$ and a Poisson random variable with mean $x\vartheta\log(n) + c$, analogously to Lemma 4.6 in [60]. This is done in terms of the *Kolmogorov distance* $\mathrm{d_K}$; see (2.20).

Corollary 3.20. *Let g_Θ and B_n be as in Theorem 3.17 and let P_ϑ be a Poisson distributed random variable with mean $x\vartheta\log(n) + c$. Then, for any fixed $0 \leq x < 1$,*

$$\mathrm{d_K}\left(B_n(x), P_\vartheta\right) = \mathcal{O}\left(\frac{n^{x-1}}{\log(n)}\right).$$

Proof. This estimate can be established with Proposition 3.1 and Corollary 3.2 in [12] with $\chi(s) = \exp((e^{is} - 1)(x\vartheta\log(n) + \gamma + c))$, $\psi_\nu(s) = 1$ and $\psi_\mu(s) = 1$. □

Another consequence of Theorem 3.17 is the following central limit result.

Corollary 3.21. *Let g_Θ and B_n be as in Theorem 3.17. Then, for any fixed $0 \leq x \leq 1$,*

$$W_n(x) := \frac{B_n(x) - x\vartheta\log(n)}{\sqrt{\vartheta\log(n)}} \xrightarrow{d} \mathcal{N}(0, x)$$

holds as $n \to \infty$, where $\mathcal{N}(0, x)$ denotes a centered Gaussian random variable with variance x.

Proof. The case $x = 1$ follows immediately from Corollary 3.14 and we can thus assume $x < 1$. We know from (3.21) that $L_{D_x}(\rho) = x\vartheta\log(n) + \mathcal{O}(1)$, as $n \to \infty$. We obtain with (3.20)

$$\begin{aligned}\mathbb{E}_\Theta\left[\exp\left(\frac{is}{\sqrt{\vartheta\log(n)}} B_n(x)\right)\right] &= \exp\left(\left(e^{is/\sqrt{\vartheta\log(n)}} - 1\right) x\vartheta\log(n)\right)(1 + o(1)) \\ &= \exp\left(isx\sqrt{\vartheta\log(n)} - \frac{s^2 x}{2}\right)(1 + o(1))\end{aligned}$$

and the result follows. □

We now turn to the main result of this section. As already mentioned, it was shown that the process W_n given in (3.19), considered with respect to the uniform measure (and with respect to the Ewens measure, when W_n is properly rescaled) converges weakly to the standard Brownian motion. In our setting, the analogous statement is the following.

Theorem 3.22. *Suppose that $g_\Theta(t)$ belongs to $\mathcal{F}(\rho, \vartheta, K)$ and define*

$$W_n(x) := \frac{B_n(x) - x\vartheta \log(n)}{\sqrt{\vartheta \log(n)}}.$$

Then, as $n \to \infty$ and for $0 \le x \le 1$, W_n converges weakly to the standard Brownian motion $\mathcal{W}$ on $[0,1]$.

Proof. We will proof this statement following the arguments of Hansen [43]. First define $\alpha(n,x) := \min\{\lfloor n^x \rfloor, n/\log^2(n)\}$ and consider the truncated process

$$\bar{B}_n(x) := \sum_{m=1}^{\alpha(n,x)} C_m.$$

Furthermore, define a process

$$\widetilde{W}_n(x) := \big(\bar{B}_n(x) - L_{D_x}(\rho)\big)/\sqrt{\vartheta \log(n)} \tag{3.22}$$

with D_x as in Theorem 3.17. It follows that

$$|W_n(x) - \widetilde{W}_n(x)| \le \frac{\sum_{m=\alpha(n,x)}^{\lfloor n^x \rfloor} C_m}{\sqrt{\vartheta \log(n)}} + \frac{|L_{D_x}(\rho) - x\vartheta \log(n)|}{\sqrt{\vartheta \log(n)}}.$$

The first term converges to zero in distribution since Markov's inequality yields

$$\mathbb{P}_\Theta\left[\sum_{m=\alpha(n,x)}^{\lfloor n^x \rfloor} C_m \ge \sqrt{\vartheta \log(n)}\right] \le \frac{1}{\vartheta \log(n)} \sum_{m=n/\log^2(n)}^{n} \mathbb{E}_\Theta\left[C_m\right] = \mathcal{O}\left(\frac{\log\log(n)}{\log(n)}\right).$$

The second term is $o(1)$ as $n \to \infty$ with $o(1)$ uniform in $x \in [0,1]$. Therefore, the distance between $W_n(x)$ and $\widetilde{W}_n(x)$ is asymptotically vanishing with respect to the Skorohod topology on the space of right-continuous functions with left limits. It is thus sufficient to prove $\widetilde{W}_n \xrightarrow{d} \mathcal{W}$. We will proceed in two steps: first, we will show that the process $\widetilde{W}_n$ converges to $\mathcal{W}$ in terms of finite-dimensional distributions and then its tightness.

Convergence of the finite dimensional distributions. We have to show that for any $k \in \mathbb{N}$ and $0 \le x_1 < x_2 < ... < x_k \le 1$ the random vector $\{\widetilde{W}_n(x_j)\}_{j=1}^k$ converges in distribution to the vector $\{\mathcal{N}(0, x_j)\}_{j=1}^k$ with independent increments. We know from Corollary 3.21 that $\widetilde{W}_n(x_j) \xrightarrow{d} \mathcal{N}(0, x_j)$ for all $x_j \in [0,1]$. It remains to show that the increments are independent. Define the sets

$$D_n^{(j)} := \{\lfloor n^{x_{j-1}} \rfloor + 1, \dots, \lfloor n^{x_j} \rfloor\}$$

with $x_0 := 0$ and a straightforward application of Lemma 2.3 gives

$$h_n \mathbb{E}_\Theta \left[e^{\sum_{j=1}^k \left(is_j (\bar{B}_n(x_j) - \bar{B}_n(x_{j-1})\right)} \right] = [t^n] \left[e^{g_\Theta(t) + \sum_{j=1}^k (e^{is_j}-1) L_{D_n^{(j)}}(t)} \right]. \tag{3.23}$$

We can thus apply Theorem 3.6. The remaining computations are the same as in the proof of Corollary 3.21. Notice that here we need the truncated process $\bar{B}_n$ so that the case $x_k = 1$ does not cause any problems when we apply Theorem 3.6.

Tightness. It remains to prove that the process $\widetilde{W}_n$ is tight. We use the moment condition given in [20, Theorem 15.6]. More precisely, we show that for any $n \geq 0$ and $0 \leq x_1 < x < x_2 \leq 1$,

$$E_\Theta^{\widetilde{W}_n} := \mathbb{E}_\Theta \left[\big(\widetilde{W}_n(x) - \widetilde{W}_n(x_1)\big)^2 \big(\widetilde{W}_n(x_2) - \widetilde{W}_n(x)\big)^2 \right] = \mathcal{O}\big((x_2 - x_1)^2\big).$$

We start with the identity

$$h_n E_\Theta^{\widetilde{W}_n} = \frac{[t^n]}{(\vartheta \log(n))^2} \Big[\Big(\big(L_{D_1}(t) - E_{x_1}^x\big)^2 + L_{D_1}(t) \Big) \Big(\big(L_{D_2}(t) - E_x^{x_2}\big)^2 + L_{D_2}(t) \Big)\ e^{g_\Theta(t)} \Big], \tag{3.24}$$

where

$$E_a^b = \mathbb{E}_\Theta \left[\bar{B}_n(b) - \bar{B}_n(a) \right]$$

and $D_1 = \{\lfloor n^{x_1} \rfloor + 1, ..., \lfloor n^x \rfloor\}$, $D_2 = \{\lfloor n^x \rfloor + 1, ..., \lfloor n^{x_2} \rfloor\}$. The proof of this formula is based on the *randomization method* (see Section 2.1) and can be found in [21, Lemma 4.10]. Now, it follows with Corollary 3.10 that

$$\begin{aligned}
&[t^n] \Big[\Big(\big(L_{D_1}(t) - E_{x_1}^x\big)^2 + L_{D_1}(t) \Big) \Big(\big(L_{D_2}(t) - E_x^{x_2}\big)^2 + L_{D_2}(t) \Big)\ e^{g_\Theta(t)} \Big] \\
=& \mathcal{O}\bigg(\frac{n^{\vartheta - 1}}{\rho^n} \Big(\big(L_{D_1}(\rho) - E_{x_1}^x\big)^2 + L_{D_1}(\rho) \Big) \Big(\big(L_{D_2}(\rho) - E_x^{x_2}\big)^2 + L_{D_2}(\rho) \Big) \bigg) \\
=& \mathcal{O}\bigg(\frac{n^{\vartheta - 1}}{\rho^n}\ L_{D_1}(\rho) L_{D_2}(\rho) \bigg).
\end{aligned}$$

The last equality holds since equation (3.21) implies

$$L_{D_1}(\rho) - E_{x_1}^x = \sum_{m = \lfloor n^{x_1} \rfloor + 1}^{\lfloor n^x \rfloor} \frac{\theta_m}{m} \rho^m - \sum_{\alpha(n, x_1)}^{\alpha(n,x)} \mathbb{E}_\Theta \left[C_m \right] = \mathcal{O}(\log\log(n))$$

and similarly

$$L_{D_2}(\rho) - E_x^{x_2} = \sum_{m = \lfloor n^{x} \rfloor + 1}^{\lfloor n^{x_2} \rfloor} \frac{\theta_m}{m} \rho^m - \sum_{\alpha(n, x)}^{\alpha(n,x_2)} \mathbb{E}_\Theta \left[C_m \right] = \mathcal{O}(\log\log(n))$$

while

$$L_{D_1}(\rho) = \mathcal{O}\big((x - x_1)\log(n)\big) \quad \text{and} \quad L_{D_2}(\rho) = \mathcal{O}\big((x_2 - x)\log(n)\big).$$

Therefore

$$E_\Theta^{\widetilde{W}_n} = \mathcal{O}\left(\frac{L_{D_1}(\rho)L_{D_2}(\rho)}{\log^2(n)}\right) = \mathcal{O}\big((x - x_1)(x_2 - x)\big) = \mathcal{O}\big((x_2 - x_1)^2\big),$$

which completes the proof the tightness. □

3.6 A functional central limit theorem - with restriction

In the last section we considered the process W_n without restriction of the probability measure, that is under the condition $A_n = \{1, \ldots, n\}$. Verifying the proof of Theorem 3.22 carefully, one notices that our argument is based on the equations (3.23) and (3.24), but they require only minor modifications when $A_n \neq \{1, \ldots, n\}$. Thus, one can apply this proof for many possible restrictions A_n, as long as the assumptions of Theorem 3.6 are satisfied. Since the argument for all the interesting cases is similar, we restrict our interest to $A_n = \{\lceil n^a \rceil, \ldots, n\}$ with $0 \leq a < 1$. In this case, the characteristic function of $B_n(x)$ for $0 \leq x < 1$ behaves like

$$\mathbb{E}_\Theta^{(A_n)}\left[\exp(isB_n(x))\right] = \exp\left((e^{is} - 1)L_{M_n}(\rho)\right)\left(1 + \mathcal{O}\left(\frac{\max\{n^x, n^a\}}{n}\right)\right),$$

where $M_n = A_n \cap \{1, \ldots, \lfloor n^x \rfloor\} = \{\lceil n^a \rceil, \ldots, \lfloor n^x \rfloor\}$. We get with (3.7)

$$L_{M_n}(\rho) = \sum_{m=1}^{\lfloor n^x \rfloor} \frac{\theta_m}{m}\rho^m = (x - a)\vartheta \log(n)\, \mathbb{1}_{\{x \geq a\}} + \mathcal{O}(1).$$

One can now use the same method as in Section 3.5 to show that, as $n \to \infty$,

$$\frac{B_n(x) - \max\{x - a, 0\}\,\vartheta\log(n)}{\sqrt{\vartheta\log(n)}} \xrightarrow{d} \mathcal{W}_a(x), \tag{3.25}$$

where $\mathcal{W}_a(x)$ is the continuous process on $[0, 1]$ with

$$\mathcal{W}_a(x) \stackrel{d}{=} \begin{cases} \mathcal{N}(0, x - a) & \text{if } x \geq a, \\ 0 & \text{otherwise.} \end{cases}$$

In other terms, for $A_n = \{\lceil n^a \rceil, \ldots, n\}$, the process defined on the left-hand side of (3.25) converges weakly to a Brownian motion started at $x = a$.

Another interesting case appears when considering functionals that only involve cycles with even or odd cycle length. Define the processes

$$B_n^{(ev)}(x) := \sum_{\substack{1 \leq m \leq n^x \\ m \text{ even}}} C_m \quad \text{and} \quad B_n^{(odd)}(x) := \sum_{\substack{1 \leq m \leq n^x \\ m \text{ odd}}} C_m.$$

Assume that we have no restrictions on the cycle lengths, that is $A_n = \{1, \ldots, n\}$ in Definition 3.1. We need to find an appropriate rescaling for $B_n^{(ev)}$ and $B_n^{(odd)}$ in order to prove joint convergence to the Brownian motion. To simplify the calculations, consider again truncated processes

$$\bar{B}_n^{(ev)}(x) := \sum_{\substack{1 \le m \le \alpha(n,x) \\ m \text{ even}}} C_m \quad \text{and} \quad \bar{B}_n^{(odd)}(x) := \sum_{\substack{1 \le m \le \alpha(n,x) \\ m \text{ odd}}} C_m$$

with $\alpha(n, x) = \min\{\lfloor n^x \rfloor, n/\log^2(n)\}$. With the same argument as in the proof of Theorem 3.22, it suffices to prove the convergence for $\bar{B}_n^{(ev)}$ and $\bar{B}_n^{(odd)}$. First, we need to compute their joint characteristic function. For $0 \le x_1, x_2 \le 1$ we have

$$\begin{aligned} &h_n \mathbb{E}_\Theta \left[\exp\left(is_1 \bar{B}_n^{(ev)}(x_1) + is_2 \bar{B}_n^{(odd)}(x_2)\right)\right] \\ =&[t^n] \left[\exp\left(g_\Theta(t) + (e^{is_1} - 1) L_{D_n^{(ev)}}(t) + (e^{is_2} - 1) L_{D_n^{(odd)}}(t)\right)\right] \end{aligned}$$

with $D_n^{(ev)} = \{m \le n^{x_1} | m \text{ even}\}$ and $D_n^{(odd)} = \{m \le n^{x_1} | m \text{ odd}\}$. This is proven with our usual approach. Then, apply Theorem 3.6 for $0 \le x_1, x_2 \le 1$ and get

$$\begin{aligned} &\mathbb{E}_\Theta \left[\exp\left(is_1 \bar{B}_n^{(ev)}(x_1) + is_2 \bar{B}_n^{(odd)}(x_2)\right)\right] \\ &= \exp\left((e^{is_1} - 1) L_{D_n^{(ev)}}(\rho)\right) \exp\left((e^{is_2} - 1) L_{D_n^{(odd)}}(\rho)\right) \left(1 + \mathcal{O}\left(\frac{\log^2(n)}{n}\right)\right). \end{aligned}$$

Notice that at this place we need the truncated processes, otherwise we could apply Theorem 3.6 only for $0 \le x_1, x_2 < 1$. With (3.7) follows that

$$L_{D_n^{(ev)}}(\rho) = x_1 \frac{\vartheta}{2} \log(n) + \mathcal{O}(1) \quad \text{and} \quad L_{D_n^{(odd)}}(\rho) = x_2 \frac{\vartheta}{2} \log(n) + \mathcal{O}(1)$$

and therefore we define the rescaled processes

$$W_n^{(ev)}(x) := \frac{B_n^{(ev)}(x) - x\frac{\vartheta}{2}\log(n)}{\sqrt{\frac{\vartheta}{2}\log(n)}} \quad \text{and} \quad W_n^{(odd)}(x) := \frac{B_n^{(odd)}(x) - x\frac{\vartheta}{2}\log(n)}{\sqrt{\frac{\vartheta}{2}\log(n)}}.$$

Our aim is to prove the following theorem.

Theorem 3.23. *The processes $W_n^{(ev)}(x)$ and $W_n^{(odd)}(x)$ converge, as $n \to \infty$, to two independent standard Brownian motions for $0 \le x \le 1$.*

Proof. First, define

$$\widetilde{W}_n^{(ev)}(x) := \frac{\bar{B}_n^{(ev)}(x) - x\frac{\vartheta}{2}\log(n)}{\sqrt{\frac{\vartheta}{2}\log(n)}} \quad \text{and} \quad \widetilde{W}_n^{(odd)}(x) := \frac{\bar{B}_n^{(odd)}(x) - x\frac{\vartheta}{2}\log(n)}{\sqrt{\frac{\vartheta}{2}\log(n)}}.$$

We will prove the required convergence for these two processes and then it also holds for $W_n^{(ev)}$ and $W_n^{(odd)}$. We proceed as in the proof of Corollary 3.21 to see that for $0 \le x_1, x_2 \le 1$, as $n \to \infty$,

$$\left(\widetilde{B}_n^{(odd)}(x_1), \widetilde{B}_n^{(odd)}(x_2)\right) \xrightarrow{d} (\mathcal{N}_1, \mathcal{N}_2)\,,$$

where $\mathcal{N}_1$ and $\mathcal{N}_2$ are independent centered Gaussian random variables with variance x_1, x_2, respectively. Notice that if we would work with $W_n^{(ev)}(x)$ and $W_n^{(odd)}(x)$ directly, then our standard argument would only be valid for $0 \le x_1, x_2 < 1$ and a more complicated argument would be needed for $x_1 = 1$ and/or $x_2 = 1$.

Finally, it remains to prove that the increments of $\widetilde{W}_n^{(ev)}$ and $\widetilde{W}_n^{(odd)}$ are independent and the tightness of both processes. The argument is similar to that in Section 3.5; see (3.23) and (3.24). $\square$

4

The order of permutations under the generalized Ewens measure

We denote as usual by $\mathfrak{S}_n$ the symmetric group of degree n. This chapter is devoted to the order of a permutation $\sigma \in \mathfrak{S}_n$, denoted by $O_n = O_n(\sigma)$, which is defined as the smallest integer $k \geq 1$ such that the k-th iterate of σ gives the identity. Landau [55] proved in 1909 that the maximum of the order of all $\sigma \in \mathfrak{S}_n$ satisfies, for $n \to \infty$, the asymptotic

$$\max_{\sigma \in \mathfrak{S}_n} \log O_n(\sigma) \sim \sqrt{n \log(n)}. \tag{4.1}$$

On the other hand, $O_n(\sigma)$ can be computed as the least common multiple of the cycle length of σ. Thus, if σ is a permutation that consists of only one cycle of length n, then $\log O_n(\sigma) = \log(n)$, and $(n-1)!$ of all $n!$ permutations share this property. Considering these two extremal types of behavior, the famous result of Erdős and Turán [32] seems even more remarkable: they showed in 1965 that a uniformly chosen random permutations satisfies, as $n \to \infty$, the Normal limit law

$$\frac{\log O_n - \frac{1}{2}\log^2(n)}{\sqrt{\frac{1}{3}\log^3(n)}} \xrightarrow{d} \mathcal{N}(0,1). \tag{4.2}$$

The original proof was direct and rather technical. Thereafter, several authors gave probabilistic proofs of this limit theorem, among them those of Best [14] in 1970, DeLaurentis and Pittel [25] in 1985 who use a functional central limit theorem for the cycle counts, and Arratia and Tavaré [9] in 1992, whose proof is based on the Feller coupling. This result was also extended to the Ewens measure and to A-permutations; see for instance [9] and [77].

In this chapter we study the random variable $\log O_n$ with respect to the weighted measure $\mathbb{P}_\Theta$ which is given in Definition 1.1: for $\Theta = (\theta_m)_{m \geq 1}$ with $\theta_m \geq 0$ for every

$m \geq 1$ we have for $\sigma \in \mathfrak{S}_n$

$$\mathbb{P}_\Theta[\sigma] := \frac{1}{h_n n!} \prod_{m=1}^{n} \theta_m^{C_m}. \tag{4.3}$$

Recall also the important identity described in Corollary 2.4:

$$\sum_{n=0}^{\infty} h_n t^n = \exp\big(g_\Theta(t)\big) \quad \text{with} \quad g_\Theta(t) = \sum_{m=1}^{\infty} \frac{\theta_m}{m} t^m. \tag{4.4}$$

The parameters we are interested in are the *generalized Ewens parameters* which were defined in Section 3.1, see Definition 3.3 and in particular the definition of the class $\mathcal{F}(\rho, \vartheta, K)$ in Definition 3.4. The extension of the Erdős-Turán law (4.2) for this model is straightforward; see Theorem 5.11. Furthermore, we establish a local limit theorem and large deviations estimates for $\log O_n$ which are, to our knowledge, new even for the uniform measure. To this end, define

$$\Omega_n := \frac{\log O_n - \frac{\vartheta}{2}\log^2(n)}{\log^{4/3}(n)}.$$

Under some mild additional conditions on the parameters $\Theta = (\theta_m)_{m \geq 1}$ we will prove in Section 4.3 the following local limit theorem:

Theorem 4.1. *Suppose that g_Θ belongs to $\mathcal{F}(\rho, \vartheta, K)$ and that $\theta_m \rho^m = \vartheta + \mathcal{O}(m^{-\delta})$ for some $\delta > 0$. Then the following holds for any bounded Borel subset $B \subset \mathbb{R}$ with boundary of Lebesgue measure zero:*

$$\lim_{n\to\infty} \sigma_n \, \mathbb{P}_\Theta[\Omega_n \in B] = \frac{m(B)}{\sqrt{2\pi}},$$

where $m(B)$ denotes the Lebesgue measure of B and $\sigma_n = \sqrt{\vartheta/3}\log^{1/6}(n)$.

Subsequently, in Section 4.4 we will prove that under some extra moment condition the following precise large deviations estimate holds for any $x > 0$:

$$\mathbb{P}_\Theta\left[\Omega_n \geq x\,\sigma_n^2\right] = \frac{\exp(-\sigma_n^2 \frac{x^2}{2} + \frac{x^3\vartheta}{18})}{\sqrt{2\pi\sigma_n^2 x^2}}\,(1 + o(1)),$$

where σ_n is as in Theorem 4.1. Finally, in Section 4.6 we present a precise expression for the expected value of $\log O_n$:

$$\begin{aligned}\mathbb{E}_\Theta[\log O_n] &= \mathbb{E}_\Theta[\log Y_n] - \vartheta\log(n)\big(1 - \log(\vartheta\log(n))\big) \\ &\quad + \sum_{\varrho} \Gamma(-\varrho)\big(\vartheta\log(n)\big)^{\varrho} + \mathcal{O}\left((\log\log(n))^3\right),\end{aligned}$$

where $\sum_\varrho$ denotes the sum over the non-trivial zeros of the Riemann zeta function. This statement has an immediate interpretation in terms of the Riemann hypothesis; see Corollary 4.27, which extends results of Zacharovas [78].

4.1 Preliminaries

Recall that the order $O_n(\sigma)$ of a permutation $\sigma \in \mathfrak{S}_n$ is the smallest integer $k \geq 1$ such that the k-th iterate of σ gives the identity. Assume that σ decomposes into disjoint cycles $\sigma_1 \cdots \sigma_\ell$ and denote by λ_i the length of cycle σ_i. Then $O_n(\sigma)$ can be computed as the least common multiple of the cycle length:

$$O_n(\sigma) = \operatorname{lcm}(\lambda_1, \lambda_2, \cdots \lambda_\ell).$$

A common approach to investigate the asymptotic behavior of $\log O_n$ is to introduce the random variable

$$Y_n := \prod_{m=1}^{n} m^{C_m}, \quad \text{that is} \quad \log Y_n = \sum_{m=1}^{n} \log(m)\, C_m, \tag{4.5}$$

where the C_m denotes as usual the number of cycles of length m. The basic strategy is to establish results for $\log Y_n$ and then to show that it is relatively close to $\log O_n$ in a certain sense. To give explicit expressions for O_n and Y_n involving the C_m let us introduce

$$D_{nk} := \sum_{m=1}^{n} C_m \mathbb{1}_{\{k|m\}} \quad \text{and} \quad D^*_{nk} := \min\{1, D_{nk}\}. \tag{4.6}$$

Now let $p_1, p_2, \ldots$ be the prime numbers and $q_{m,i}$ be the multiplicity of a prime number p_i in the number m. Then

$$Y_n = \prod_{m=1}^{n} (p_1^{q_{m,1}} p_2^{q_{m,2}} \cdots p_n^{q_{m,n}})^{C_m} = \prod_{i=1}^{n} p_i^{C_1 \cdot q_{1,i} + C_2 \cdot q_{2,i} + \cdots + C_n \cdot q_{n,i}} = \prod_{p \leq n} p^{\sum_{j=1}^{n} D_{np^j}}, \tag{4.7}$$

where $\prod_{p \leq n}$ denotes the product over all prime numbers that are less or equal to n. The last equality can be understood as follows: first, notice that $D_{nk} = 0$ for $k > n$. Next, let p be fixed and define $m = p^{q_{m,i}} \cdot a$ where a and p are coprime (meaning that their least common divisor is 1). Then C_m appears exactly once in the sum D_{np^j} if $j \leq q_{m,i}$ but it does not appear if $j > q_{m,i}$. Thus, C_m appears $q_{m,i}$ times in the sum $\sum_{j=1}^{n} D_{np^j}$. Analogously, we have

$$O_n = \prod_{p \leq n} p^{\sum_{j=1}^{n} D^*_{np^j}}. \tag{4.8}$$

To simplify the logarithm of the expressions (4.7) and (4.8), we introduce the von Mangoldt function Λ, which is defined as

$$\Lambda(n) = \begin{cases} \log(p) & \text{if } n = p^k \text{ for some prime } p \text{ and } k \geq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{4.9}$$

Consequently,

$$\log Y_n = \sum_{k\le n} \Lambda(k) D_{nk} \quad \text{and} \quad \log O_n = \sum_{k\le n} \Lambda(k) D^*_{nk}. \tag{4.10}$$

Now define

$$\Delta_n \coloneqq \log Y_n - \log O_n = \sum_{k\le n} \Lambda(k)\big(D_{nk} - D^*_{nk}\big). \tag{4.11}$$

In order to prove properties of $\log O_n$ they are first established for $\log Y_n$ and then one needs to show that Δ_n is approximately small enough to transfer the result to $\log O_n$; see for example Lemma 4.4. An important tool to study $\log Y_n$ is its generating function. Using the *randomization method* we showed in Section 2.1 that

$$\sum_{n=0}^{\infty} h_n \mathbb{E}_\Theta[\exp(s \log Y_n)] t^n = \exp\left(\sum_{m=1}^{\infty} \frac{\theta_m}{m^{1-s}} t^m\right) \tag{4.12}$$

holds; see (2.7). This and other quantities of interest will be studied with respect to the *generalized Ewens measure* and with the same singularity analysis tools as in Chapter 3. For the readers convenience, let us recall the relevant definitions.

Definition 4.2. *Let $0 < \rho < R$ and $0 < \phi < \frac{\pi}{2}$ be given. We then define*

$$\Delta_0 \coloneqq \Delta_0(\rho, R, \phi) \coloneqq \{t \in \mathbb{C}; |t| < R, t \neq \rho, |\arg(t-\rho)| > \phi\}.$$

See an Illustration of the Δ_0-domain in Figure 3. The parameters we are interested in are those which have a generating function g_Θ which is analytic in a Δ_0-domain and admits logarithmic growth at its dominant singularity.

Definition 4.3. *Let $\rho, \vartheta > 0$ and $K \in \mathbb{R}$ be given. We write $\mathcal{F}(\rho, \vartheta, K)$ for the set of all functions g satisfying the following two conditions:*

(1) g is holomorphic in $\Delta_0(\rho, R, \phi)$ for some $R > \rho$ and $0 < \phi < \frac{\pi}{2}$,

(2)

$$g(t) = \vartheta \log\left(\frac{1}{1 - t/\rho}\right) + K + \mathcal{O}(t-\rho) \qquad \text{as } t \to \rho \text{ for } t \in \Delta_0. \tag{4.13}$$

We also recall the statement of Remark 3.5, which gives a justification of the name *generalized Ewens measure*: let g_Θ be as in (4.4) and let the parameters θ_m be such that g_Θ belongs to $\mathcal{F}(\rho, \vartheta, K)$. Then there exists some ϵ_m such that

$$\theta_m \rho^m = \vartheta + \epsilon_m \quad \text{with } \epsilon_m \to 0 \text{ and } \sum_{m=1}^{\infty} \frac{|\epsilon_m|}{m} < \infty. \tag{4.14}$$

Finally, recall that Corollary 3.8 with $D_n = \emptyset$ gives us the behavior of h_n for this model:

$$h_n = \frac{n^{\vartheta-1}e^K}{\rho^n\Gamma(\vartheta)}\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right). \tag{4.15}$$

The starting point of our study of the properties of $\log O_n$ is the closeness of $\log O_n$ and $\log Y_n$. Recall the definition of Δ_n in (4.11).

Lemma 4.4. *Let $(\theta_m)_{m\geq 1}$ be such that g_Θ belongs to $\mathcal{F}(\rho,\vartheta,K)$. Then, as $n\to\infty$, the following asymptotic holds for every constant κ:*

$$\mathbb{P}_\Theta\left[\Delta_n \geq \log(n)(\log\log(n))^\kappa\right] = \mathcal{O}\big((\log\log(n))^{1-\kappa}\big).$$

The analogous result for the Ewens measure was proved in [25]. In Section 4.6, a much more precise expression for $\mathbb{E}_\Theta\left[\Delta_n\right]$ is presented. For the proof of Lemma 4.4 the following proposition is required.

Proposition 4.5. *Suppose that g_Θ belongs to $\mathcal{F}(\rho,\vartheta,K)$. Then*

(1) $\mathbb{E}_\Theta\left[D_{nk}\right] = \mathcal{O}\left(\dfrac{\log(n)}{k} + n^{-\theta}\,\mathbb{1}_{\{k|n\}}\right)$,

(2) $\mathbb{E}_\Theta\left[D_{nk}(D_{nk}-1)\right] = \mathcal{O}\left(\dfrac{\log^2(n)}{k^2} + n^{-2\theta}\,\mathbb{1}_{\{k|n\}}\right)$.

Furthermore, the error terms are uniform in k for $1\leq k\leq n$.

Proof of Lemma 4.4. Notice that Δ_n defined in (4.11) can be estimated as

$$\Delta_n = \sum_{k=1}^{n}\Lambda(k)\big(D_{nk}-D_{nk}^*\big) =: \sum_{k=1}^{n}\Lambda(k)\,\Delta_{nk}$$

with

$$\Delta_{nk}\leq D_{nk} \quad\text{and}\quad \Delta_{nk}\leq D_{nk}(D_{nk}-1).$$

Thus

$$\mathbb{E}_\Theta\left[\Delta_n\right] = \sum_{k=1}^{n}\Lambda(k)\,\mathbb{E}_\Theta[\Delta_{nk}] \leq \sum_{k=1}^{\lfloor\log(n)\rfloor}\Lambda(k)\,\mathbb{E}_\Theta[D_{nk}] + \sum_{k=\lceil\log(n)\rceil}^{n}\Lambda(k)\,\mathbb{E}_\Theta[D_{nk}(D_{nk}-1)].$$

Then Proposition 4.5 together with (2.26) and (2.27) gives

$$\mathbb{E}_\Theta\left[\Delta_n\right] = \mathcal{O}\Bigg(\log(n)\sum_{k=1}^{\lfloor\log(n)\rfloor}\frac{\Lambda(k)}{k} + \log^2(n)\sum_{k=\lceil\log(n)\rceil}^{n}\frac{\Lambda(k)}{k^2}\Bigg) = \mathcal{O}(\log(n)\log\log(n)). \tag{4.16}$$

Now Chebychev's inequality implies for $n \to \infty$

$$\mathbb{P}_\Theta\left[\Delta_n \geq \log(n)(\log\log(n))^\kappa\right] \leq \frac{\mathbb{E}_\Theta[\Delta_n]}{\log(n)(\log\log(n))^\kappa} = \mathcal{O}\big((\log\log(n))^{1-\kappa}\big)$$

and this completes the proof of the lemma. □

Proof of Proposition 4.5 . We begin with (1). Lemma 2.5 and (4.6) yield

$$\mathbb{E}_\Theta\left[D_{nk}\right] = \sum_{m=1}^{n} \mathbb{E}_\Theta\left[C_m\right] \mathbb{1}_{\{k|m\}} = \sum_{m=1}^{n} \frac{\theta_m}{m} \mathbb{1}_{\{k|m\}} \frac{h_{n-m}}{h_n}.$$

We have to distinguish the cases $\vartheta \geq 1$ and $\vartheta < 1$; see (4.14). If $\vartheta \geq 1$, then (4.14) and (4.15) imply that $\theta_m h_{n-m}/h_n$ is bounded and thus

$$\mathbb{E}_\Theta\left[D_{nk}\right] = \mathcal{O}\left(\sum_{m=1}^{n} \frac{1}{m} \mathbb{1}_{\{k|m\}}\right) = \mathcal{O}\left(\frac{1}{k}\sum_{j=1}^{n/k}\frac{1}{j}\right) = \mathcal{O}\left(\frac{\log(n)}{k}\right).$$

If $\vartheta < 1$, we have to be more careful. Again, (4.14) and (4.15) yield

$$\mathbb{E}_\Theta\left[D_{nk}\right] = \mathcal{O}\left(\sum_{m=1}^{n-1} \frac{1}{m} \mathbb{1}_{\{k|m\}} \left(1-\frac{m}{n}\right)^{\theta-1} + n^{-\theta}\, \mathbb{1}_{\{k|n\}}\right),$$

where

$$\begin{aligned}
\sum_{m=1}^{n-1} \frac{1}{m} \mathbb{1}_{\{k|m\}} \left(1-\frac{m}{n}\right)^{\theta-1} &= \mathcal{O}\left(\sum_{m=1}^{n/2} \frac{\mathbb{1}_{\{k|m\}}}{m} + \frac{1}{n}\sum_{m>n/2}^{n-1} \mathbb{1}_{\{k|m\}}\left(1-\frac{m}{n}\right)^{\theta-1}\right)\\
&= \mathcal{O}\left(\frac{\log(n)}{k} + \frac{1}{n}\int_{n/(2k)}^{(n-1)/k}\left(1-\frac{kx}{n}\right)^{\theta-1} dx\right)\\
&= \mathcal{O}\left(\frac{\log(n)}{k} + \frac{1}{k}\right) = \mathcal{O}\left(\frac{\log(n)}{k}\right).
\end{aligned}$$

This completes the proof of (1). Furthermore,

$$\begin{aligned}
\mathbb{E}_\Theta\left[D_{nk}(D_{nk}-1)\right] &= \mathbb{E}_\Theta\left[\left(\sum_{m=1}^{n} C_m \mathbb{1}_{\{k|m\}}\right)\left(\sum_{m=1}^{n} C_m \mathbb{1}_{\{k|m\}} - 1\right)\right]\\
&= \mathbb{E}_\Theta\left[\sum_{m,m'=1}^{n} C_m C_{m'} \mathbb{1}_{\{k|m\,;\,k|m'\}} - \sum_{m=1}^{n} C_m \mathbb{1}_{\{k|m\}}\right]\\
&= \mathbb{E}_\Theta\left[\sum_{m,m'=1;m\neq m'}^{n} C_m C_{m'} \mathbb{1}_{\{k|m\,;\,k|m'\}} + \sum_{m=1}^{n} C_m(C_m-1) \mathbb{1}_{\{k|m\}}\right]\\
&= \sum_{\substack{m,m'=1\\ m\neq m'}}^{n} \frac{\theta_m}{m}\frac{\theta_{m'}}{m'} \mathbb{1}_{\{k|m\,;\,k|m'\}} \frac{h_{n-m-m'}}{h_n} + \sum_{m=1}^{n}\left(\frac{\theta_m}{m}\right)^2 \mathbb{1}_{\{k|m\}} \frac{h_{n-2m}}{h_n}.
\end{aligned}$$

A similar argument as for $\mathbb{E}_\Theta\left[D_{nk}\right]$ gives the upper bound in (2). □

With Lemma 4.4 at hand, one can directly deduce the Erdős-Turán law as it was stated in (4.2) for uniform random permutations.

Theorem 4.6. *Suppose that g_Θ belongs to $\mathcal{F}(\rho, \vartheta, K)$. Then, as $n \to \infty$, one has*

$$\frac{\log O_n - \frac{\vartheta}{2}\log^2(n)}{\sqrt{\frac{\vartheta}{3}\log^3(n)}} \xrightarrow{d} \mathcal{N}(0,1).$$

Proof. Given Lemma 4.4, it suffices to show the required asymptotic for $\log Y_n$. In a beautiful proof, DeLaurentis and Pittel [25] deduce this for the uniform measure from a functional version of the central limit theorem for the cycle counts. For our model, the proof is very similar, but we present it here for the sake of completeness. Consider the functionals

$$B_n(x) := \sum_{m=1}^{\lfloor n^x \rfloor} C_m \qquad \text{and} \qquad W_n(x) := \frac{B_n(x) - x\vartheta\log(n)}{\sqrt{\vartheta\log(n)}}$$

and recall that we proved in Theorem 3.22 that, as $n \to \infty$ and for $0 \le x \le 1$, the process W_n converges weakly to the standard Brownian motion $\mathcal{W}$ on $[0,1]$. Set $t_{m/n} = \log(m)/\log(n)$ and notice that

$$W_n(t_{m/n}) = \frac{B_n(m) - \vartheta\log(m)}{\sqrt{\vartheta\log(n)}}.$$

This gives the following telescope sum

$$\begin{aligned}\vartheta\log Y_n &= (\vartheta\log(n))^2\Big(1 - \sum_{m=1}^{n-1} t_{m/n}(t_{m+1/n} - t_{m/n})\Big)\\ &\quad + (\vartheta\log(n))^{3/2}\Big(W_n(1) - \sum_{m=1}^{n-1} W_n(t_{m/n})(t_{m+1/n} - t_{m/n})\Big).\end{aligned}$$

Now notice that the functional $W_n(\cdot)$ is constant on the intervals $[t_{m/n}, t_{m+1/n})$ and that $\log(m+1) - \log(m) = \log(1+1/m) = 1/m + \mathcal{O}(1/m^2)$. Then, the Euler-Maclaurin formula (2.33) yields

$$\vartheta\log Y_n = (\vartheta\log(n))^2\Big(\frac{1}{2} + \mathcal{O}\Big(\log^{-2}(n)\Big)\Big) + (\vartheta\log(n))^{3/2}\Big(W_n(1) - \int_0^1 W_n(t)dt\Big).$$

Hence, by Theorem 3.22,

$$\frac{\vartheta\log Y_n - \frac{1}{2}(\vartheta\log(n))^2}{(\vartheta\log(n))^{3/2}} \xrightarrow{d} \int_0^1 W(1) - W(t)dt \stackrel{d}{=} \int_0^1 W(t)dt \stackrel{d}{=} \mathcal{N}\Big(0, \frac{1}{3}\Big)$$

and the proof is completed.

□

4.2 The truncated order

To establish further properties of the order of permutations, it turns out to be convenient to introduce truncated versions of $\log Y_n$ and $\log O_n$ in order to simplify computations. Therefore, define

$$\widetilde{O}_n = \operatorname{lcm}\{m \le b_n;\, C_m \neq 0\} \quad \text{with} \quad b_n := n/\log^2(n) \tag{4.17}$$

and similarly

$$\widetilde{Y}_n := \prod_{m=1}^{b_n} m^{C_m}.$$

The advantage of the truncated variables is that less analytic assumptions on the generating function g_Θ are required and that many computations are simpler; see also Remark 4.9. Nonetheless, $\widetilde{Y}_n$ and $\widetilde{O}_n$ share many important properties with Y_n and O_n. Similarly to (4.10) we have

$$\log \widetilde{Y}_n = \sum_{k\le n} \Lambda(k)\widetilde{D}_{nk} \quad \text{with} \quad \widetilde{D}_{nk} := \sum_{m=1}^{b_n} C_m\, \mathbb{1}_{\{k|m\}}, \tag{4.18}$$

$$\log \widetilde{O}_n = \sum_{k\le n} \Lambda(k)\widetilde{D}^*_{nk} \quad \text{with} \quad \widetilde{D}^*_{nk} := \min\{1, \widetilde{D}_{nk}\}. \tag{4.19}$$

Our basic strategy is as follows: we will establish properties of $\log \widetilde{Y}_n$ and transfer them to $\log \widetilde{O}_n$ and finally to $\log O_n$. For the first transfer, define

$$\widetilde{\Delta}_n := \log \widetilde{Y}_n - \log \widetilde{O}_n$$

and notice that $0 \le \widetilde{\Delta}_n \le \Delta_n$. Thus, Lemma 4.4 yields

$$\mathbb{P}_\Theta\left[\widetilde{\Delta}_n \ge \log(n)(\log\log(n))^\kappa\right] = \mathcal{O}\big((\log\log(n))^{1-\kappa}\big). \tag{4.20}$$

For the second transfer, notice that

$$\log O_n - \log \widetilde{O}_n \le \log Y_n - \log \widetilde{Y}_n = \sum_{m=b_n+1}^{n} \log(m)\, C_m = \mathcal{O}\big(\log(n)\log\log(n)\big).$$

In order to study $\log \widetilde{Y}_n$, we need its moment generating function.

Lemma 4.7. *Let $g_\Theta(t)$ be as in (4.4) and $s \in \mathbb{C}$, then*

(1) $\mathbb{E}_\Theta\left[\log \widetilde{Y}_n\right] = \dfrac{1}{h_n}[t^n]\left[\left(\sum_{m=1}^{b_n} \log(m)\dfrac{\theta_m}{m}t^m\right)\exp\left(g_\Theta(t)\right)\right],$

(2) $\mathbb{E}_\Theta\left[e^{s\log \widetilde{Y}_n}\right] = \dfrac{1}{h_n}[t^n]\left[\exp\left(g_\Theta(t) + \left(\sum_{m=1}^{b_n}(e^{s\log(m)}-1)\dfrac{\theta_m}{m}t^m\right)\right)\right],$

where the functions on the right-hand sides are considered as formal power series in t.

Proof. Equation (1) follows from (2) by differentiating once with respect to s and substituting $s=0$. We thus only have to prove (2). For this, let $c \in \mathbb{N}$ be fixed and consider $Y_n^c := \prod_{m=1}^{c} m^{C_m}$. We apply Lemma 2.3 with $a_m = e^{s\log(m)}\theta_m$ for $m \le c$ and $a_m = \theta_m$ for $m > c$. We then have as formal power series

$$\begin{aligned}\sum_{n=0}^{\infty} h_n t^n \mathbb{E}_\Theta\left[e^{s\log Y_n^c}\right] &= \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{\sigma\in\mathfrak{S}_n} \prod_{m=1}^{c} (e^{s\log(m)}\theta_m)^{C_m} \prod_{m=c+1}^{\infty} \theta_m^{C_m} \\ &= \exp\left(\sum_{m=1}^{c} e^{s\log(m)}\frac{\theta_m}{m}t^m + \sum_{m=c+1}^{\infty}\frac{\theta_m}{m}t^m\right) \\ &= \exp\left(g_\Theta(t) + \left(\sum_{m=1}^{c}(e^{s\log(m)}-1)\frac{\theta_m}{m}t^m\right)\right).\end{aligned}$$

Now identify the coefficients of t^n on both sides and obtain

$$\mathbb{E}_\Theta\left[e^{s\log Y_n^c}\right] = \frac{1}{h_n}[t^n]\left[\exp\left(g_\Theta(t) + \left(\sum_{m=1}^{c}(e^{s\log(m)}-1)\frac{\theta_m}{m}t^m\right)\right)\right].$$

Equation (2) follows by substituting $c = b_n$. □

The previous lemma yields

Lemma 4.8. *If g_Θ belongs to $\mathcal{F}(\rho,\vartheta,K)$, then*

$$\mathbb{E}_\Theta\left[\log \widetilde{Y}_n\right] = \sum_{m=1}^{b_n} \frac{\log(m)}{m}\theta_m\rho^m + \mathcal{O}\left(\log^{-1}(n)\right).$$

Furthermore, we get for $s \in \mathbb{C}$

$$\mathbb{E}_\Theta\left[e^{s\frac{\log \widetilde{Y}_n}{\log(n)}}\right] = \exp\left(\sum_{m=1}^{b_n}\left(e^{\frac{s\log(m)}{\log(n)}}-1\right)\frac{\theta_m}{m}\rho^m\right)\left(1+\mathcal{O}\left(n^{-1}\right)\right)$$

and the error term is uniform in s for s bounded.

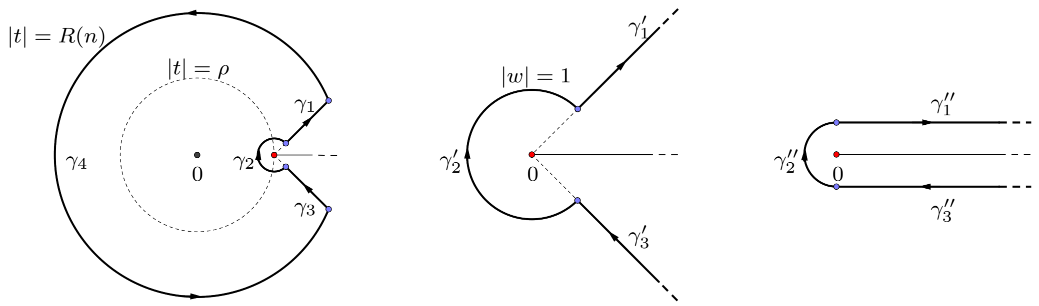
Proof. We use Lemma 4.7 and get with Cauchy's integral formula (see Theorem 2.11)

$$\begin{aligned} h_n\mathbb{E}_\Theta\left[\log \widetilde{Y}_n\right] &= \frac{1}{2\pi i}\int_\gamma \widetilde{q}_1(t)\exp\left(g_\Theta(t)\right)\frac{dt}{t^{n+1}}, \\ h_n\mathbb{E}_\Theta\left[e^{s\frac{\log \widetilde{Y}_n}{\log(n)}}\right] &= \frac{1}{2\pi i}\int_\gamma \widetilde{e}(s,t)\exp\left(g_\Theta(t)\right)\frac{dt}{t^{n+1}},\end{aligned}$$

where γ is a simple closed curve around 0 and

$$\widetilde{q}_1(t) := \sum_{m=1}^{b_n} \log(m)\frac{\theta_m}{m}t^m, \qquad \widetilde{e}(s,t) := \sum_{m=1}^{b_n} \left(e^{s\frac{\log(m)}{\log(n)}} - 1\right)\frac{\theta_m}{m}t^m.$$

By assumption, g_Θ is analytic in a domain $\Delta_0 = \Delta(\rho, R, \phi)$; see Definition 4.2 and Definition 4.3. We choose for both integrals the same curve γ as the proof of Theorem 3.6, see Figure 4, such that γ is contained in the Δ_0-domain. More



(a) $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$ (b) $\gamma' = \gamma_1' \cup \gamma_2' \cup \gamma_3'$ (c) $\gamma'' = \gamma_1'' \cup \gamma_2'' \cup \gamma_3''$

Figure 5: The curves used in the proof of Lemma 4.8.

precisely, the radius of the big circle γ_4 is $R(n) := \rho(1+b_n^{-1})$ with b_n as in (4.17), the radius of the small circle is $1/n$ and the angle of the line segments is independent of n. Notice that $\widetilde{q}_1(t)$ and $e(s,t)$ are for given n polynomials and we thus do not require any further analytic assumptions to use this curve. First, we consider the integral over γ_4 and show that its contribution is negligible. We get with (4.14) and for $\varphi \in [-\pi, \pi]$

$$\begin{aligned}\left|\widetilde{q}_1(R(n)e^{i\varphi})\right| &= \mathcal{O}\bigg(\sum_{m=1}^{b_n} \frac{\log(m)}{m}\left(1+b_n^{-1}\right)^m\bigg) = \mathcal{O}\bigg(\sum_{m=1}^{b_n} \frac{\log(m)}{m}\left(1+\mathcal{O}\left(mb_n^{-1}\right)\right)\bigg)\\ &= \mathcal{O}\bigg(\sum_{m=1}^{b_n} \frac{\log(m)}{m}\bigg) = \mathcal{O}\big(\log^2(b_n)\big) = \mathcal{O}\big(\log^2(n)\big).\end{aligned}$$

We have used that $m \le b_n$ and thus $m\log(1+b_n^{-1}) = \frac{m}{b_n}(1+o(1))$. Since $(e^{s\frac{\log(m)}{\log(n)}} - 1)$ is bounded for s bounded, we can apply for $\widetilde{e}(s,t)$ the same estimate as for $\widetilde{q}_1$ and get

$$\left|\widetilde{e}(s, R(n)e^{i\varphi})\right| = \mathcal{O}\big(\log^2(b_n)\big) = \mathcal{O}\big(\log^2(n)\big).$$

Furthermore, we have on the domain Δ_0

$$|g_\Theta(t)| \le \vartheta \log\left|\frac{1}{1-t/\rho}\right| + \mathcal{O}(1) \quad\Longrightarrow\quad |g_\Theta(R(n)e^{i\varphi})| \le \vartheta\log(b_n) + \mathcal{O}(1).$$

Finally,

$$\begin{aligned}R(n)^{-n} &= \rho^{-n}\big(1+n^{-1}\log^2(n)\big)^{-n} = \rho^{-n}\exp\big(-\log^2(n)+\mathcal{O}(\log^4(n)/n)\big)\\ &= \mathcal{O}\big(\rho^{-n}\exp(-\log^2(n))\big).\end{aligned}$$

Combining these three estimates yields

$$\left|\frac{1}{2\pi i}\int_{\gamma_4}\widetilde{q}_1(t)\exp\left(g_\Theta(t)\right)\frac{dt}{t^{n+1}}\right| = \mathcal{O}\big(\rho^{-n}n^{\vartheta}\exp(-\log^2(n))\big).$$

Since $h_n \sim e^K n^{\vartheta-1}\rho^{-n}\Gamma^{-1}(\vartheta)$ (see (4.15)), we can neglect the integral over γ_4 with respect to the scale of the problem. Let us consider the remaining parts of the curve. The computations of the integrals over γ_1,γ_2 and γ_3 are completely similar to the computations in the proof of Theorem VI.3 in [39] (the idea was also sketched in Section 2.2). We thus give only a short overview. We start with $\widetilde{q}_1$ and write $t=\rho(1+wn^{-1})$ with $w=\mathcal{O}(\log^2(n))$ and obtain

$$\begin{aligned}\widetilde{q}_1\left(\rho+\frac{rw}{n}\right) &= \sum_{m=1}^{b_n}\frac{\log(m)}{m}\theta_m\rho^m\left(1+\frac{w}{n}\right)^m = \sum_{m=1}^{b_n}\frac{\log(m)}{m}\theta_m\rho^m\left(1+\mathcal{O}\left(\frac{mw}{n}\right)\right)\\ &= \sum_{m=1}^{b_n}\frac{\log(m)}{m}\theta_m\rho^m + \mathcal{O}\left(\frac{w}{n}\sum_{m=1}^{b_n}\log(m)\right)\\ &= \sum_{m=1}^{b_n}\frac{\log(m)}{m}\theta_m\rho^m + \mathcal{O}\left(\frac{w}{\log(n)}\right).\end{aligned}\tag{4.21}$$

We now use the asymptotic behavior of $g_\Theta(t)$ at ρ in (4.13) to get

$$\begin{aligned}&\frac{1}{2\pi i}\int_{\gamma_1\cup\gamma_2\cup\gamma_3}\widetilde{q}_1(t)\exp\left(g_\Theta(t)\right)\frac{dt}{t^{n+1}}\\ =&\frac{n^{\vartheta-1}}{2\pi i\rho^n}e^K\int_{\gamma'}\widetilde{q}_1\left(\rho+\frac{\rho w}{n}\right)(-w)^{-\vartheta}e^{-w}(1+\mathcal{O}(w/n))\,dw\\ =&\frac{n^{\vartheta-1}}{2\pi i\rho^n}e^K\left(\sum_{m=1}^{b_n}\frac{\log(m)}{m}\theta_m\rho^m\int_{\gamma'}(-w)^{-\vartheta}e^{-w}dw+\mathcal{O}\left(\log^{-1}(n)\right)\right),\end{aligned}\tag{4.22}$$

where γ' is the bounded curve in Figure 5(b). We have used for the estimate of the error term that $\mathrm{Re}(e^{-w})$ is decreasing exponentially fast as $\mathrm{Re}(w)\to\infty$. Furthermore, a simple contour argument allows us to replace the bounded curve γ' by the infinite Hankel contour γ'' as in Figure 5(c). Notice that

$$\frac{1}{2\pi i}\int_{\gamma''}(-w)^{-\vartheta}e^{-w}\,dw = \frac{1}{\Gamma(\vartheta)},\tag{4.23}$$

where $\vartheta\in\mathbb{C}$ is arbitrary (details can be found for instance in [39, Section B.3]). Combining (4.23) with (4.22) and Corollary 3.8 completes the proof of the first assertion. The argument for the second is very similar. One only has to replace (4.21) by

$$\widetilde{e}\left(s, \rho + \frac{\rho w}{n}\right) = \sum_{m=1}^{b_n} \left(e^{s\frac{\log(m)}{\log(n)}} - 1\right)\frac{\theta_m}{m}\rho^m + \mathcal{O}\left(\frac{w}{n}\right).$$

□

Remark 4.9. Instead of the truncated sequence $\log \widetilde{Y}_n$ one may consider the generating functions for $\log Y_n$ which are given by

$$\mathbb{E}_\Theta\left[\log Y_n\right] = \frac{1}{h_n}[t^n]\left[q_1(t)\exp\left(g_\Theta(t)\right)\right] \text{ and } \mathbb{E}_\Theta\left[e^{s\log Y_n}\right] = \frac{1}{h_n}[t^n]\left[\exp\big(e(s,t) + g_\Theta(t)\big)\right]$$

with

$$q_1(t) := \left(\sum_{m=1}^{\infty} \log(m)\frac{\theta_m}{m}t^m\right) \text{ and } e(s,t) = \sum_{m=1}^{\infty}(e^{s\log(m)} - 1)\frac{\theta_m}{m}t^m.$$

To use the same contour as in the proof of Lemma 4.8, analytic extensions of $q_1(t)$ and $e(s,t)$ to some Δ_0-domain plus the asymptotic behavior at ρ are required. However, for all probabilistic question we consider here, except the precise expected value of $\log O_n$ in Section 4.6, it is enough to know the behavior of the truncated variables $\log \widetilde{Y}_n$ since they are transferable to $\log Y_n$.

Remark 4.10. To simplify certain computations, we will assume in some cases

$$\theta_m \rho^m = \vartheta + \mathcal{O}(m^{-\delta})$$

for some $\delta > 0$. Then the Euler Summation formula (2.34) yields

$$\sum_{m=1}^{b_n} \frac{\log(m)}{m}\theta_m\rho^m = \vartheta\sum_{m=1}^{b_n}\frac{\log(m)}{m} + \mathcal{O}(1) = \frac{\vartheta}{2}\log^2(b_n) + \mathcal{O}(1), \tag{4.24}$$

$$\sum_{m=1}^{b_n} \frac{\theta_m}{m}\rho^m \mathbb{1}_{\{k|m\}} = \vartheta\frac{\log(b_n)}{k} + \mathcal{O}\left(\frac{\log(k)}{k}\right).$$

With this assumption, we get a nice expression for the moment generating function of $\log \widetilde{Y}_n$.

Corollary 4.11. *If g_Θ belongs to $\mathcal{F}(\rho, \vartheta, K)$ and $\theta_m\rho^m = \vartheta + \mathcal{O}(m^{-\delta})$ for some $\delta > 0$, then*

$$\mathbb{E}_\Theta\left[e^{s\frac{\log \widetilde{Y}_n}{\log(n)}}\right] = \exp\left(\log(b_n)\left(\frac{e^s}{s} - \frac{1}{s} - 1\right) + \mathcal{O}\left(\frac{s}{\log(n)}\right)\right)\left(1 + \mathcal{O}(n^{-1})\right).$$

Proof. Corollary 4.11 follows immediately from Lemma 4.8 and a simple application of the Euler summation formula (2.34). □

4.3 A local limit theorem

Given the characteristic function of $\log \widetilde{Y}_n$ in Lemma 4.8, we prove in this section Theorem 4.1, which says that the local behavior of the rescaled order of a permutation is well-controlled. To this end, define

$$\widetilde{\mathcal{Y}}_n := \frac{\log \widetilde{Y}_n - \frac{\vartheta}{2}\log^2(n)}{\log^{4/3}(n)}.$$

It will turn out that $\widetilde{\mathcal{Y}}_n$ satisfies the so-called mod-Gaussian convergence; see Definition 2.17. We will then apply Theorem 2.21 to get a local limit theorem for $\widetilde{\mathcal{Y}}_n$ which we will transfer to

$$\Omega_n := \frac{\log O_n - \frac{\vartheta}{2}\log^2(n)}{\log^{4/3}(n)}. \tag{4.25}$$

More precisely, for $\sigma_n = \sqrt{\vartheta/3}\log^{1/6}(n)$, we will show that for any bounded Borel subset $B \subset \mathbb{R}$ with boundary of Lebesgue measure zero

$$\lim_{n\to\infty} \sigma_n \, \mathbb{P}_\Theta\left[\Omega_n \in B\right] = \frac{m(B)}{\sqrt{2\pi}}$$

holds, where $m(B)$ denotes the Lebesgue measure of B. To prove this, let us first show that $\widetilde{\mathcal{Y}}_n$ is indeed mod-Gaussian convergent in Lemma 4.12. Subsequently, we present in Lemma 4.13 that $\widetilde{\mathcal{Y}}_n$ satifies the required local behavior. Finally, the result has to be transferred to Ω_n.

Lemma 4.12. *Suppose that g_Θ belongs to $\mathcal{F}(\rho, \vartheta, K)$ and $\theta_m \rho^m = \vartheta + \mathcal{O}(m^{-\delta})$ for some $\delta > 0$. The sequence $\widetilde{\mathcal{Y}}_n$ is mod-$\mathcal{N}(0, \sigma_n^2)$ convergent with $\sigma_n^2 = \frac{\vartheta}{3}\log^{1/3}(n)$ and limiting function given by $\Phi(x) = e^{x^3\vartheta/18}$.*

Proof. Take the generating function in Lemma 4.8 and expand the exponential term to get

$$\begin{aligned}\mathbb{E}_\Theta\left[e^{s\frac{\log \widetilde{Y}_n}{\log n}}\right] &= \exp\Big(s\frac{\vartheta}{2}\big(\log(n) + \mathcal{O}(\log\log(n)\big) + \frac{s^2}{2}\frac{\vartheta}{3}\big(\log(n) + \mathcal{O}(\log\log(n)\big) \\ &\quad + \frac{s^3}{3!}\frac{\vartheta}{4}\big(\log(n) + \mathcal{O}(\log\log(n)\big) + \mathcal{O}(s^4\log(n))\Big)\left(1 + \mathcal{O}(n^{-1})\right);\end{aligned}$$

we used (4.24) and similar estimates for the higher order terms. Since $s \in \mathbb{C}$ we may write $s = it$ with $t \in \mathbb{R}$ and get

$$\mathbb{E}_\Theta\left[e^{it\frac{\log \widetilde{Y}_n}{\log^{4/3} n}}\right] = \exp\Big(it\frac{\vartheta}{2}\log^{\frac{2}{3}}(n) - \frac{t^2}{2}\frac{\vartheta}{3}\log^{\frac{1}{3}}(n) + \frac{it^3}{3!}\frac{\vartheta}{4} + \mathcal{O}\Big(\frac{\log\log(n)}{\log^{1/3}(n)}\Big)\Big)$$

and this gives the result. $\square$

As a direct consequence, we get a local limit theorem for $\widetilde{\mathcal{Y}}_n$.

Lemma 4.13. *Under the conditions of Lemma 4.12 the following holds for any bounded Borel subset $B \subset \mathbb{R}$ with boundary of Lebesgue measure zero:*

$$\lim_{n\to\infty} \sigma_n \,\mathbb{P}_\Theta \left[\widetilde{\mathcal{Y}}_n \in B\right] = \frac{m(B)}{\sqrt{2\pi}},$$

where $m(B)$ denotes the Lebesgue measure of B and σ_n is defined as in Lemma 4.12.

Proof. Apply Theorem 2.21 with $\varphi(t) = e^{-t^2/2}$ and σ_n as in Lemma 4.12. We need to verify that condition **H3** holds, that is we have to show the uniform integrability of the sequence

$$f_{nk} := \mathbb{E}_\Theta\left[e^{it\frac{\widetilde{\mathcal{Y}}_n}{\sigma_n}}\right] \mathbb{1}_{|t\sigma_n^{-1}|\leq k}$$

for all $k \geq 0$. Set $\bar{t} := t\sigma_n^{-1}$ and recall that Lemma 4.12 implies

$$\begin{aligned}
\mathbb{E}_\Theta\left[e^{i\bar{t}\widetilde{\mathcal{Y}}_n}\right] &= \exp\Big(-\bar{t}^2\sigma_n^2 + \frac{\vartheta\, i\bar{t}^3}{18} + \mathcal{O}\Big(\bar{t}^4 \frac{\log\log(n)}{\log^{1/3}(n)}\Big)\Big)\\
&= \exp\Big(-t^2 + \frac{\vartheta}{18}\frac{it^3}{\sigma_n^3} + \mathcal{O}\Big(t^4\frac{\log\log(n)}{\log(n)}\Big)\Big).
\end{aligned}$$

Thus

$$\left|\mathbb{E}_\Theta\left[e^{i\bar{t}\widetilde{\mathcal{Y}}_n}\right]\right| = \exp\big(-t^2 + o(1)\big)$$

which implies the uniform integrability. $\square$

Proof of Theorem 4.1. It remains to transfer the result from $\widetilde{\mathcal{Y}}_n$ to

$$\widetilde{\Omega}_n := \frac{\log \widetilde{O}_n - \frac{\vartheta}{2}\log^2(n)}{\log^{4/3}(n)}$$

and subsequently to Ω_n defined in Theorem 4.1. To this end, notice that for every $\epsilon > 0$ there exist Jordan-measurable sets (meaning that they are bounded with boundary of Lebesgue measure zero) $B_\epsilon \subset B \subset B^\epsilon$ such that

$$m(B^\epsilon \setminus B) \leq \epsilon \quad \text{ and } \quad m(B \setminus B_\epsilon) \leq \epsilon.$$

To see this, notice that ∂B is bounded (since B is bounded) and that it is also closed (complement of the interior and the exterior, both open sets), thus ∂B is compact. Cover ∂B with open rectangles whose total volume does not exceed ϵ. Since ∂B is compact, U can be chosen to be a finite union of open rectangles. Then define

$$B_\epsilon := B \setminus U \quad \text{ and } \quad B^\epsilon := B \cup U$$

to get the required sets (they are indeed Jordan-measurable since $\partial(B\backslash U) \subset \partial B \cup \partial U$ and $\partial(B \cup U) \subset \partial B \cup \partial U$). This gives

$$\mathbb{P}_\Theta\left[\widetilde{\Omega}_n \in B\right] \leq \mathbb{P}_\Theta\left[\widetilde{\mathcal{Y}}_n \in B^\epsilon\right] + \mathcal{O}\Big(\mathbb{P}_\Theta\left[\log \widetilde{Y}_n - \log \widetilde{O}_n \geq \epsilon \log^{4/3}(n)\right]\Big)$$

and

$$\mathbb{P}_\Theta\left[\widetilde{\Omega}_n \in B\right] \geq \mathbb{P}_\Theta\left[\widetilde{\mathcal{Y}}_n \in B^\epsilon\right] + \mathcal{O}\Big(\mathbb{P}_\Theta\left[\log \widetilde{Y}_n - \log \widetilde{O}_n \geq \epsilon \log^{4/3}(n)\right]\Big).$$

Thus, we have to show

$$\sigma_n \mathbb{P}_\Theta\left[\log \widetilde{Y}_n - \log \widetilde{O}_n \geq \epsilon \log^{4/3}(n)\right] \to 0 \qquad \text{as } n \to \infty. \tag{4.26}$$

This is true since

$$\mathbb{P}_\Theta\left[\log \widetilde{Y}_n - \log \widetilde{O}_n \geq \epsilon \log^{4/3}(n)\right] \leq \mathbb{P}_\Theta\left[\log Y_n - \log O_n \geq \epsilon \log^{4/3}(n)\right]$$

and then (4.16) and Markov's inequality yield the required asymptotic. Now (4.26) implies

$$\lim_{n\to\infty} \sigma_n \mathbb{P}_\Theta\left[\widetilde{\Omega}_n \in B\right] \leq \lim_{n\to\infty} \sigma_n \mathbb{P}_\Theta\left[\widetilde{\mathcal{Y}}_n \in B^\epsilon\right] = \frac{m(B^\epsilon)}{\sqrt{2\pi}} \leq \frac{m(B)+\epsilon}{\sqrt{2\pi}}.$$

With the same argument for the reversed inequality, we get that for all $\epsilon > 0$,

$$\frac{m(B)-\epsilon}{\sqrt{2\pi}} \leq \lim_{n\to\infty} \sigma_n \mathbb{P}_\Theta\left[\widetilde{\Omega}_n \in B\right] \leq \frac{m(B)+\epsilon}{\sqrt{2\pi}}.$$

Let ϵ tend to zero to obtain

$$\lim_{n\to\infty} \sigma_n \, \mathbb{P}_\Theta\left[\widetilde{\Omega}_n \in B\right] = \frac{m(B)}{\sqrt{2\pi}}.$$

With the same argument, the result is transferred from $\widetilde{\Omega}_n$ to Ω_n, assuming that

$$\sigma_n \mathbb{P}_\Theta\left[\log O_n - \log \widetilde{O}_n \geq \epsilon \log^{4/3}(n)\right] \to 0$$

is satisfied as $n \to \infty$. To see this, notice that

$$\mathbb{P}_\Theta\left[\log O_n - \log \widetilde{O}_n \geq \epsilon \log^{4/3}(n)\right] \leq \mathbb{P}_\Theta\left[\log Y_n - \log \widetilde{Y}_n \geq \epsilon \log^{4/3}(n)\right]$$

holds as well as

$$\mathbb{E}_\Theta\left[\log Y_n - \log \widetilde{Y}_n\right] = \mathcal{O}\big(\log(n) \log\log(n)\big).$$

□

4.4 Large deviations estimates

This section is devoted to two large deviations estimates for $\log O_n$. To our knowledge, these results are new even for the uniform measure. The first estimate is established by a classical large deviations approach. We will show in Theorem 4.15 that for any Borel set B

$$\limsup_{n\to\infty} \frac{1}{\log(n)} \log \mathbb{P}_\Theta\left(\frac{\log O_n}{\log^2(n)} \in B\right) = -\inf_{x\in B} F(x) \tag{4.27}$$

where

$$F(x) := \sup_{t\in\mathbb{R}}[tx - \chi(t)]$$

is the so-called Fenchel-Legendre transform of $\chi(t) := \frac{e^t-1-t}{t}$. This result was stated by O'Connell [61] for the uniform measure. However, his proof of Lemma 2 is incorrect. Here, we give a detailed proof based on an extra moment condition and even present a refined result, namely a precise large deviations estimate; see Theorem 4.17.

Let us first discuss the moment condition which we need to establish our results.

Moment condition Let g_Θ belong to $\mathcal{F}(\rho, \vartheta, K)$ and assume $\theta_m\rho^m = \vartheta + \mathcal{O}(m^{-\delta})$ for some $\delta > 0$. Define

$$\Delta_{n,\beta(n)} := \sum_{k=\beta(n)}^{n} \Lambda(k)(\widetilde{D}_{nk} - \widetilde{D}^*_{nk}),$$

where $\beta(n) = \exp(\log^x(n))$ for some $x < 1$. Then the moment condition is satisfied if for any $m \in \mathbb{N}$ the following holds:

$$\mathbb{E}_\Theta\left[(\Delta_{n,\beta(n)})^m\right] = \mathcal{O}_m\big((\log(n)\log\log(n))^m\big). \tag{4.28}$$

Remark 4.14. We are strongly convinced that the moment condition is satisfied under the above assumptions, however we are so far not able to prove it. The condition is clearly satisfied for $m = 1$ and for $m = 2$ the reader can find a proof in the appendix.

With the moment generating function of $\log \widetilde{Y}_n/\log(n)$ stated in Corollary 4.11 at hand, a simple application of the Gärtner-Ellis Theorem yields an estimate as in (4.27) for $\log \widetilde{Y}_n$. Then, using the moment condition (4.28), we show by exponential equivalence that this estimate can be transferred to $\log \widetilde{O}_n$ and then to $\log O_n$. More precisely, we will prove the following

Theorem 4.15. *Assume that g_Θ belongs to $\mathcal{F}(\rho, \vartheta, K)$ with $\theta_m\rho^m = \vartheta + \mathcal{O}(m^{-\delta})$ for some $\delta > 0$ and that the moment condition* (4.28) *holds. Then the sequence $\log O_n/\log^2(n)$ satisfies a large deviations principle with rate $\log(n)$ and rate function given by the Fenchel-Legendre transform of $\chi(t) := \frac{e^t-1-t}{t}$.*

Proof. Let us first check that $\log \widetilde{Y}_n / \log^2(n)$ satisfies the required large deviations principle. By the Gärtner-Ellis Theorem, it suffices to check

$$\lim_{n\to\infty} \frac{1}{\log(n)} \log \mathbb{E}_\Theta \left[\exp\left(t \frac{\log \widetilde{Y}_n}{\log(n)} \right) \right] = \chi(t)$$

and this follows immediately from Corollary 4.11. Proving exponential equivalence, Lemma 4.16 transfers this result from $\log \widetilde{Y}_n$ to $\log \widetilde{O}_n$ and then to $\log O_n$. □

Lemma 4.16. *Under the assumptions of Theorem 4.15 the following holds for any* $c > 0$*:*

(1) $\displaystyle\limsup_{n\to\infty} \frac{1}{\log(n)} \log \mathbb{P}_\Theta \left[\log \widetilde{Y}_n - \log \widetilde{O}_n > c \log^2(n) \right] = -\infty,$

(2) $\displaystyle\limsup_{n\to\infty} \frac{1}{\log(n)} \log \mathbb{P}_\Theta \left[\log O_n - \log \widetilde{O}_n > c \log^2(n) \right] = -\infty.$

Proof. We will prove stronger versions of (1) and (2) in Lemma 4.18 and Lemma 4.19 below. □

The result of Theorem 4.15 can be even refined.

Theorem 4.17. *Define* $\sigma_n^2 = \frac{\vartheta}{3} \log^{1/3}(n)$. *Under the assumptions of Theorem 4.15 the following holds for any* $x > 0$,

$$\mathbb{P}_\Theta \left[\Omega_n \geq x\, \sigma_n^2 \right] = \frac{\exp(-\sigma_n^2 \frac{x^2}{2} + \frac{x^3 \vartheta}{18})}{\sqrt{2\pi \sigma_n^2 x^2}} (1 + o(1)).$$

To prove this result, we proceed as follows: from the mod-Gaussian convergence of $\widetilde{\mathcal{Y}}_n$ stated in Lemma 4.12 we deduce a precise large deviations estimate for $\widetilde{\mathcal{Y}}_n$ by using Theorem 2.22. Subsequently, using the moment condition (4.28) we prove exponential equivalence similar to Lemma 4.16 to transfer the estimate first to $\widetilde{\Omega}_n$ and then so Ω_n.

Proof of Theorem 4.17. First, combine Lemma 4.12 with Theorem 2.22 for $\beta_n = \sigma_n^2$, $F(x) = x^2/2 = \eta(x)$ and Φ as in Lemma 4.12. This gives the required precise deviations estimate for $\widetilde{\mathcal{Y}}_n$. Let us first transfer this result to $\widetilde{\Omega}_n$. Clearly,

$$\mathbb{P}_\Theta \left[\widetilde{\Omega}_n \geq x\, \sigma_n^2 \right] \leq \mathbb{P}_\Theta \left[\widetilde{\mathcal{Y}}_n \geq x\, \sigma_n^2 \right].$$

For the other direction, let g be a function such that $g(n) = o(\sigma_n^2)$. Then

$$\mathbb{P}_\Theta \left[\widetilde{\mathcal{Y}}_n \geq x\, \sigma_n^2 + g(n) \right] \leq \mathbb{P}_\Theta \left[\widetilde{\Omega}_n \geq x\, \sigma_n^2 \right] + \mathbb{P}_\Theta \left[\log Y_n - \log \widetilde{O}_n \geq g(n) \log^{4/3}(n) \right]$$

holds as well as

$$\mathbb{P}_\Theta \left[\widetilde{\mathcal{Y}}_n \geq x\, \sigma_n^2 + g(n) \right] = \mathbb{P}_\Theta \left[\widetilde{\mathcal{Y}}_n \geq x\, \sigma_n^2 \right] \big(1 + o(1)\big).$$

Therefore, we have to prove

$$\lim_{n\to\infty} \frac{1}{\sigma_n^2} \log \mathbb{P}_\Theta \left[\log Y_n - \log \widetilde{O}_n \geq g(n) \log^{4/3}(n)\right] = -\infty$$

and this is the subject of Lemma 4.19. Using the same argument, Lemma 4.18 transfers the result to Ω_n. □

Lemma 4.18. *Under the assumptions of Theorem 4.17 the following holds for any* $c > 0$:

$$\lim_{n\to\infty} \frac{1}{\sigma_n^2} \log \mathbb{P}_\Theta \left[\log O_n - \log \widetilde{O}_n > c \log^{4/3}(n)\right] = -\infty.$$

Proof. We have

$$\log O_n - \log \widetilde{O}_n \leq \log Y_n - \log \widetilde{Y}_n$$

and thus the assertion is proved if we can show

$$\lim_{n\to\infty} \frac{1}{\log^{1/3}(n)} \log \mathbb{P}_\Theta \left[\log Y_n - \log \widetilde{Y}_n > c \log^{4/3}(n)\right] = -\infty.$$

Notice that

$$\log Y_n - \log \widetilde{Y}_n = \sum_{m=b_n+1}^{n} \log(m)\, C_m \leq \log(n)\, T(b_n, n)$$

where

$$T(b_n, n) = \sum_{m=b_n+1}^{n} C_m.$$

Thus it suffices to show

$$\lim_{n\to\infty} \frac{1}{\log^{1/3}(n)} \log \mathbb{P}_\Theta \left[T(b_n, n) > c \log^{1/3}(n)\right] = -\infty.$$

With Markov's inequality we get

$$\frac{1}{\log^{1/3}(n)} \log \mathbb{P}_\Theta \Big(e^{sT(b_n,n)} \geq e^{sc\log^{1/3}(n)}\Big) \leq -sc + \frac{\log \mathbb{E}_\Theta \left[e^{sT(b_n,n)}\right]}{\log^{1/3}(n)}. \tag{4.29}$$

The generating function of $T(b_n, n)$ is given by

$$\log \mathbb{E}_\Theta \left[e^{sT(b_n,n)}\right] = \vartheta(e^s - 1)\log(n) + (K - L_{b_n}(\rho))(e^s - 1) + o(1)$$

where

$$L_{b_n}(\rho) = \sum_{m=1}^{b_n} \frac{\theta_m}{m} \rho^m = \vartheta \sum_{m=1}^{b_n} \frac{1}{m} + \mathcal{O}(1) = \vartheta \log(b_n) + \mathcal{O}(1);$$

see Theorem 3.13 with $A_n = \{1, \ldots, b_n\}$. Thus

$$(4.29) \leq -sc + \mathcal{O}\Big(\frac{e^s \log\log(n)}{\log^{1/3}(n)}\Big)$$

and choose $s = \log\log\log(n)$ to get the result. $\square$

Lemma 4.19. *Under the assumptions of Theorem 4.17 the following holds for any $c > 0$:*

$$\lim_{n\to\infty} \frac{1}{\sigma_n^2} \log \mathbb{P}_\Theta \left[\log \widetilde{Y}_n - \log \widetilde{O}_n > c \log^{4/3}(n)\right] = -\infty.$$

Proof. Notice that for any sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ in $(0,\infty)$ and any function $g(n) \to \infty$ the following holds:

$$\limsup_{n\to\infty} \frac{\log(a_n + b_n)}{g(n)} = \max\Big\{\limsup_{n\to\infty} \frac{\log(a_n)}{g(n)}, \limsup_{n\to\infty} \frac{\log(b_n)}{g(n)}\Big\}.$$

We want to find the largest $\beta(n)$ such that

$$\frac{1}{\log^{1/3}(n)} \log \mathbb{P}_\Theta \left[\sum_{m=1}^{\beta(n)} \log(m)\, C_m > \frac{c}{2} \log^{4/3}(n)\right] \to -\infty \tag{4.30}$$

is satisfied. Subsequently, by means of the moment condition (4.28) we show

$$\frac{1}{\log^{1/3}(n)} \log \mathbb{P}_\Theta \left[\sum_{k=\beta(n)}^{n} \Lambda(k)(\widetilde{D}_{nk} - \widetilde{D}^*_{nk}) > \frac{c}{2} \log^{4/3}(n)\right] \to -\infty. \tag{4.31}$$

For any $s > 0$, Markov's inequality yields

$$\begin{aligned}
&\frac{1}{\log^{1/3}(n)} \log \mathbb{P}_\Theta \left[\sum_{m=1}^{\beta(n)} \log(m)\, C_m > \frac{c}{2} \log^{4/3}(n)\right] \\
&= \frac{1}{\log^{1/3}(n)} \log \mathbb{P}_\Theta \left[\exp\Big(s \sum_{m=1}^{\beta(n)} \log(m)\, C_m\Big) > \exp\Big(\frac{sc}{2} \log^{4/3}(n)\Big)\right] \\
&\leq -\frac{sc}{2} \log(n) + \frac{1}{\log^{1/3}(n)} \log \mathbb{E}_\Theta \left[\exp\Big(s \sum_{m=1}^{\beta(n)} \log(m)\, C_m\Big)\right]. \qquad (4.32)
\end{aligned}$$

Furthermore,

$$\mathbb{E}_\Theta \left[\exp\Big(s \sum_{m=1}^{\beta(n)} \log(m)\, C_m\Big)\right] = \prod_{m=1}^{\beta(n)} \mathbb{E}^t_\Theta \left[e^{s \log(m)\, Z_m}\right] = \prod_{m=1}^{\beta(n)} e^{\frac{\vartheta}{m}(e^{s\log(m)}-1)}$$

and therefore

$$\frac{1}{\log^{1/3}(n)}\log\mathbb{E}_\Theta\left[\exp\left(s\sum_{m=1}^{\beta(n)}\log(m)\,C_m\right)\right]=\frac{\vartheta}{\log^{1/3}(n)}\sum_{m=1}^{\beta(n)}\frac{e^{s\log(m)}-1}{m}.$$

Now set $s:=\log\log(n)/\log(n)$, then for all $\beta(n)=o(n)$,

$$\frac{\vartheta}{\log^{1/3}(n)}\sum_{m=1}^{\beta(n)}\frac{e^{s\log(m)}-1}{m}=\mathcal{O}\bigg(\frac{\log\log(n)}{\log^{4/3}(n)}\sum_{m=1}^{\beta(n)}\frac{\log(m)}{m}\bigg)=\mathcal{O}\bigg(\frac{\log\log(n)\log^2(\beta(n))}{\log^{4/3}(n)}\bigg).$$

Thus, set $\beta(n):=\exp(\sqrt{\log(n)})$ to obtain

$$(4.32)=-\frac{c\log\log(n)}{2}+\mathcal{O}\bigg(\frac{\log\log(n)}{\log^{1/3}(n)}\bigg)$$

and therefore assertion (4.30) is proved. So let us consider (4.31). Again, for $s>0$, and with the notation from the moment condition (4.28),

$$\begin{aligned}\frac{1}{\log^{1/3}(n)}\log\mathbb{P}_\Theta\left[\Delta_{n,\beta(n)}>\frac{c}{2}\log^{4/3}(n)\right]&=\frac{1}{\log^{1/3}(n)}\log\mathbb{P}_\Theta\left[e^{s\Delta_{n,\beta(n)}}>e^{\frac{sc}{2}\log^{4/3}(n)}\right]\\&\le-\frac{sc}{2}\log(n)+\frac{1}{\log^{1/3}(n)}\log\mathbb{E}_\Theta\left[e^{s\Delta_{n,\beta(n)}}\right].\end{aligned}$$

Thus, we set again $s:=\log\log(n)/\log(n)$. Define the event

$$A:=\Big\{\Delta_{n,\beta(n)}\le\frac{\log^{4/3}(n)}{\log\log(n)}\Big\}.$$

Then for $s=\log\log(n)/\log(n)$

$$\begin{aligned}\mathbb{E}_\Theta\left[e^{s\Delta_{n,\beta(n)}}\right]&=\mathbb{E}_\Theta\left[e^{s\Delta_{n,\beta(n)}}\,\mathbb{1}_{\{A\}}\right]+\mathbb{E}_\Theta\left[e^{s\Delta_{n,\beta(n)}}\,\mathbb{1}_{\{A^c\}}\right]\\&\le e^{\log^{1/3}(n)}+\mathbb{E}_\Theta\left[e^{s\Delta_{n,\beta(n)}}\,\mathbb{1}_{\{A^c\}}\right].\end{aligned}$$

We will show that

$$\frac{1}{\log^{1/3}(n)}\log\mathbb{E}_\Theta\left[e^{s\Delta_{n,\beta(n)}}\,\mathbb{1}_{\{A^c\}}\right]=\mathcal{O}(1) \tag{4.33}$$

holds. Cauchy's inequality yields

$$\begin{aligned}\mathbb{E}_\Theta\left[e^{s\Delta_{n,\beta(n)}}\,\mathbb{1}_{\{A^c\}}\right]&\le\mathbb{P}_\Theta\left[A^c\right]^2\mathbb{E}_\Theta\left[e^{2s\Delta_{n,\beta(n)}}\right]\\&\le\mathbb{P}_\Theta\left[A^c\right]\sum_{m=0}^{g(n)}\frac{\mathbb{E}_\Theta\left[(2s\Delta_{n,\beta(n)})^m\right]}{m!}+\sum_{m=g(n)+1}^{\infty}\frac{\mathbb{E}_\Theta\left[(2s\Delta_{n,\beta(n)})^m\right]}{m!},\end{aligned}$$

where $g(n)$ is a function to be determined in a moment. By the moment condition (4.28) and by Stirling's formula we have for $s = \log\log(n)/\log(n)$

$$\sum_{m=g(n)+1}^{\infty} \frac{\mathbb{E}_\Theta\left[(2s\Delta_{n,\beta(n)})^m\right]}{m!} = \mathcal{O}\bigg(\sum_{m=g(n)+1}^{\infty} \exp\big(m\log(2s\log(n)\log\log(n)) - m\log(m)\big)\bigg)$$
$$= \mathcal{O}\bigg(\sum_{m=g(n)+1}^{\infty} \exp\big(m\log(2(\log\log(n))^2) - m\log(m)\big)\bigg).$$

Consequently, for $g(n) = (\log\log(n))^3$, this sum satisfies (4.33). On the other hand, by Markov's inequality

$$\mathbb{P}_\Theta\left[A^c\right] = \mathbb{P}_\Theta\left[\Delta_{n,\beta(n)} > \frac{\log^{4/3}(n)}{\log\log(n)}\right] \leq \frac{\log\log(n)}{\log^{4/3}(n)}\mathbb{E}_\Theta\left[\Delta_{n,\beta(n)}\right].$$

Notice that

$$\widetilde{D}_{nk} - \widetilde{D}^*_{nk} \leq D_{nk} - D^*_{nk} \leq D_{nk}(D_{nk}-1)$$

and recall that $\beta(n) = \exp(\sqrt{\log(n)})$. Furthermore, recall (2.27) and Proposition 4.5. Then

$$\mathbb{E}_\Theta\left[\Delta_{n,\beta(n)}\right] \leq \sum_{k=\beta(n)}^{n} \Lambda(k)\mathbb{E}_\Theta\left[D_{nk}(D_{nk}-1)\right] = \mathcal{O}\bigg(\log^2(n)\sum_{k=\beta(n)}^{n}\frac{\Lambda(k)}{k^2}\bigg)$$
$$= \mathcal{O}\bigg(\frac{\log^2(n)}{\beta(n)}\bigg) = \mathcal{O}\bigg(\log^2(n)\, e^{-\sqrt{\log(n)}}\bigg).$$

This implies

$$\mathbb{P}_\Theta\left[A^c\right] = \mathcal{O}\bigg(\log^{2/3}(n)\log\log(n)\, e^{-\sqrt{\log(n)}}\bigg)$$

and therefore

$$\mathbb{P}_\Theta\left[A^c\right]\sum_{m=0}^{(\log\log(n))^3} \frac{\mathbb{E}_\Theta\left[(2s\Delta_{n,\beta(n)})^m\right]}{m!}$$
$$= \mathcal{O}\bigg(\mathbb{P}_\Theta\left[A^c\right]\sum_{m=0}^{(\log\log(n))^3} \exp\big(m\log(2(\log\log(n))^2)\big)\bigg)$$
$$= \mathcal{O}\bigg(\log^{2/3}(n)(\log\log(n))^4\exp\big(-\sqrt{\log(n)} + 2(\log\log(n))^3\log\log\log(n)\big)\bigg).$$

Altogether, we proved (4.33) and thus (4.31) holds. The proof is complete. □

4.5 The expected value of a truncated order

Recall the definition of the truncated order $\widetilde{O}_n$ in (4.17) . We will prove the following precise asymptotic expansion of $\mathbb{E}_\Theta[\log \widetilde{O}_n]$:

Theorem 4.20. *Suppose that g_Θ belongs to $\mathcal{F}(\rho, \vartheta, K)$. Then*

$$\mathbb{E}_\Theta\left[\log \widetilde{O}_n\right] = \sum_{m=1}^{b_n} \frac{\log(m)}{m}\theta_m \rho^m - \sum_{k=1}^{\log^2(n)} \Lambda(k) \exp\left(-\sum_{m=1}^{b_n} \frac{\theta_m}{m}\rho^m \mathbb{1}_{\{k|m\}}\right) - \sum_{k=1}^{\log^2(n)} \Lambda(k)\left(\sum_{m=1}^{b_n} \frac{\theta_m}{m}\rho^m \mathbb{1}_{\{k|m\}} - 1\right) + \mathcal{O}(1). \tag{4.34}$$

Before we prove this theorem, we point out the following two direct consequences.

Corollary 4.21. *Suppose that g_Θ belongs to $\mathcal{F}(\rho, \vartheta, K)$ and $\theta_m \rho^m = \vartheta + \mathcal{O}(m^{-\delta})$ for some $\delta > 0$. Then*

$$\mathbb{E}_\Theta\left[\log \widetilde{O}_n\right] = \frac{\vartheta}{2}\log^2(b_n) + \vartheta \log(b_n)\big(\log(\vartheta \log(b_n)) - 1\big) + \sum_{\varrho} \Gamma(-\varrho)(\vartheta \log(b_n))^{\varrho} + \mathcal{O}\left((\log\log(n))^3\right),$$

where $\sum_\varrho$ indicates the sum over the non-trivial zeros ϱ of Riemann zeta function.

Assuming the Riemann hypothesis to be true, that is all the non-trivial zeros of the zeta function have the form $\varrho = 1/2 + it$, any sum $\sum_\varrho x^\varrho$ with $x \geq 0$ can be estimated as $\mathcal{O}(\sqrt{x})$. This leads to the implication (1) $\Rightarrow$ (2) in the following Corollary. Moreover, similar as for the Chebychev function (2.25), we notice that the reverse implication is also true: if there would exist a zero of the zeta function of the form $\varrho = 1/2+\delta+it$ with $\delta > 0$, then we can deduce a contradiction for $\epsilon = \delta/2$. For more details we refer to the proof of (2.25) in [72, Section II.4, Corollary 3.1].

Corollary 4.22. *Suppose that $g_\Theta \in \mathcal{F}(\rho, \vartheta, K)$ and $\theta_m \rho^m = \vartheta + \mathcal{O}(m^{-\delta})$ for some $\delta > 0$. Then the following statements are equivalent*

(1) The Riemann hypothesis is true.

(2) We have for all $\epsilon > 0$

$$\mathbb{E}_\Theta\left[\log \widetilde{O}_n\right] = \frac{\vartheta}{2}\log^2(b_n) + \vartheta \log(b_n)\big(\log(\vartheta \log(b_n)) - 1\big) + \mathcal{O}\left(\log(b_n)^{1/2+\epsilon}\right).$$

Now let us deduce Corollary 4.21 from Theorem 4.20.

Proof of Corollary 4.21. Recall the estimates in Remark 4.10. Then

$$\mathbb{E}_\Theta\left[\log \widetilde{O}_n\right] = \frac{\vartheta}{2}\log^2(b_n) - \sum_{k=1}^{\log^2(n)} \Lambda(k)\left(e^{-\vartheta\frac{\log(b_n)}{k}} - 1 + \vartheta\frac{\log(b_n)}{k}\right) + \mathcal{O}\left(\frac{\log(k)}{k}\right).$$

Since $\Lambda(k) \leq \log(k)$, the sum over the error term is of order

$$\sum_{k=1}^{\log^2(n)} \Lambda(k)\mathcal{O}\left(\frac{\log(k)}{k}\right) = \mathcal{O}\left(\sum_{k=1}^{\log^2(n)} \frac{\log^2(k)}{k}\right) = \mathcal{O}\left((\log\log(n))^3\right)$$

and thus can be neglected with respect to the scale of the problem. Now consider the sum

$$\sum_{k=1}^{\log^2(n)} \Lambda(k)(e^{-x_k} - 1 + x_k) \quad \text{ with } x_k := \frac{\vartheta}{k}\log(b_n).$$

Since $e^{-x} - 1 + x = \mathcal{O}(x^2)$ as $x \to 0$, (2.27) yields

$$\sum_{k=1}^{\log^2(n)} \Lambda(k)(e^{-x_k} - 1 + x_k) = \sum_{k=1}^{\infty} \Lambda(k)(e^{-x_k} - 1 + x_k) + \mathcal{O}(1).$$

Recall that the Mellin transform of the function e^{-x} is $\Gamma(s)$ for $\mathrm{Re}(s) > 0$. Then the inverse Mellin transform gives

$$e^{-x} - 1 + x = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(s)x^{-s}\, ds \tag{4.35}$$

for $-2 < c < -1$. Details on the Mellin transform can be found for instance in [29], but here we will only need (4.35). Then

$$\sum_{k=1}^{\infty} \Lambda(k)(e^{-x_k} - 1 + x_k) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(s)\big(\vartheta\log(b_n)\big)^{-s}\sum_{k=1}^{\infty}\Lambda(k)k^s\, ds.$$

We need to justify the change of the order of summation and integration. Notice that on the line of integration

$$\left|\big(\vartheta\log(b_n)\big)^{-s}\sum_{k=1}^{\infty}\Lambda(k)k^s\right| \leq \big(\vartheta\log(b_n)\big)^{-c}\sum_{k=1}^{\infty}\Lambda(k)k^c < \infty$$

holds and thus the change of order is valid by dominated convergence. Denote by $\sum_p$ the sum over all prime numbers. It then follows by the definition of the von Mangoldt function Λ, see (4.9), that we have for $\mathrm{Re}(s) < -1$

$$\sum_{k=1}^{\infty}\Lambda(k)k^s = \sum_p \log(p)\sum_{j=1}^{\infty} p^{js} = \sum_p \log(p)\frac{p^s}{1-p^s} = -\frac{\zeta'(-s)}{\zeta(-s)}, \tag{4.36}$$

where $\zeta(s)$ denotes the Riemann zeta function. The last equality can easily be deduced form the Euler product formula of $\zeta(s)$. Therefore,

$$\sum_{k=1}^{\infty} \Lambda(k)(e^{-x_k} - 1 + x_k) = -\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s) \big(\vartheta \log(b_n)\big)^{-s} \frac{\zeta'(-s)}{\zeta(-s)} \, ds.$$

Apply now the residue theorem (see Theorem 2.10) to shift the line of integration to $1/2 + iy$ with $y \in \mathbb{R}$, which gives a double pole at $s = -1$ and simple pole at $s = 0$ and at the zeros ϱ of the zeta function. This yields

$$\sum_{k=1}^{\infty} \Lambda(k)(e^{-x_k} - 1 + x_k) = \vartheta \log(b_n)\big(1 - \log(\vartheta \log(b_n))\big) - \sum_{\varrho} \Gamma(-\varrho)(\vartheta \log(b_n))^{\varrho} - \log(2\pi) + \mathcal{O}\left((\log(b_n))^{-\frac{1}{2}}\right)$$

and the proof is complete. □

It remains to prove Theorem 4.20. Recall that $\log \widetilde{O}_n = \log \widetilde{Y}_n - \widetilde{\Delta}_n$ and that $\mathbb{E}_\Theta[\log \widetilde{Y}_n]$ was computed in Lemma 4.8. Unfortunately, the estimate given in (4.20) is not strong enough to deduce Theorem 4.20, so that we need to compute $\mathbb{E}_\Theta[\widetilde{\Delta}_n]$ more precisely. We need to study the behavior of $\widetilde{D}_{nk}$ and $\widetilde{D}^*_{nk}$, which are defined in (4.18) and (4.19).

Lemma 4.23. *For $k \in \mathbb{N}$ and $u \in \mathbb{C}$ the following holds:*

(1) $\mathbb{E}_\Theta\left[u^{\widetilde{D}_{nk}}\right] = \frac{1}{h_n}[t^n]\left[\exp\left(g_\Theta(t) + (u-1)\widetilde{g}_{\Theta,k}(t)\right)\right],$

(2) $\mathbb{E}_\Theta\left[\widetilde{D}_{nk}\right] = \frac{1}{h_n}[t^n]\left[\widetilde{g}_{\Theta,k}(t)\exp\left(g_\Theta(t)\right)\right],$

(3) $\mathbb{P}_\Theta\left[\widetilde{D}^*_{nk} = 0\right] = \frac{1}{h_n}[t^n]\left[\exp\left(g_\Theta(t) - \widetilde{g}_{\Theta,k}(t)\right)\right],$

where

$$\widetilde{g}_{\Theta,k}(t) = \sum_{m=1}^{b_n} \frac{\theta_m}{m} \mathbb{1}_{\{k|m\}} t^m. \tag{4.37}$$

Proof. Assertion (2) follows from (1) by differentiation with respect to u and substituting $u = 0$ and (3) by substituting $u = 0$ in (1). Equation (1) follows with a similar computation as in the proof of Lemma 4.7. Polya's enumeration theorem (see Lemma 2.3) together with the definition of $g_\Theta(t)$ in (4.4) and $g_{\Theta,k}(t)$ in (4.37) yields

$$\begin{aligned}\sum_{n=0}^{\infty} h_n \mathbb{E}_\Theta\left[u^{\widetilde{D}_{nk}}\right] t^n &= \sum_{n=0}^{\infty} h_n \mathbb{E}_\Theta\left[u^{\sum_{m=1}^{c} C_m \mathbb{1}_{\{k|m\}}}\right] t^n \\ &= \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{\sigma\in\mathfrak{S}_n} \prod_{\substack{m=1\\k|m}}^{c} (u\theta_m)^{C_m} \prod_{\substack{m=c+1\\k|m}}^{\infty} (\theta_m)^{C_m} \prod_{\substack{m=1\\k\nmid m}}^{c} (\theta_m)^{C_m} \\ &= \exp\left(\sum_{\substack{m=1\\k|m}}^{c} u\frac{\theta_m}{m}t^m + \sum_{\substack{m=c+1\\k|m}}^{\infty} \frac{\theta_m}{m}t^m + \sum_{\substack{m=1\\k\nmid m}}^{c} \frac{\theta_m}{m}t^m\right) \\ &= \exp\left(g_\Theta(t) + (u-1)g_{\Theta,k}(t)\right).\end{aligned}$$

Identify the coefficients of t^n on both sides and substitute $c = b_n$ to get the result. □

The previous lemma implies

Lemma 4.24. *Let g_Θ belong to $\mathcal{F}(\rho, \vartheta, K)$. We then have for $2 \le k \le n$*

(1) $\mathbb{E}_\Theta\left[\widetilde{D}_{nk}\right] = \sum_{m=1}^{b_n} \frac{\theta_m}{m}\rho^m \mathbb{1}_{\{k|m\}} + \mathcal{O}\Big(\frac{b_n}{nk}\Big),$

(2) $\mathbb{P}_\Theta\left[\widetilde{D}^*_{nk} = 0\right] = \exp\left(-\sum_{m=1}^{b_n} \frac{\theta_m}{m}\rho^m \mathbb{1}_{\{k|m\}}\right) + \mathcal{O}\Big(\frac{b_n}{nk}\Big).$

Proof. For $b_n < k \le n$ we have $\widetilde{D}_{nk} \equiv \widetilde{D}^*_{nk} \equiv 0$ and thus equations (1) and (2) are valid. So let us consider $2 \le k \le b_n$. The proof is very similar to the proof of Lemma 4.8, including the contour of integration. One only has to replace $\widetilde{q}_1(t)$ by $\widetilde{g}_{\Theta,k}(t)$ and to use

$$\widetilde{g}_{\Theta,k}\left(1 + \frac{w}{n}\right) = \sum_{m=1}^{b_n} \frac{\theta_m}{m}\rho^m \mathbb{1}_{\{k|m\}} + \mathcal{O}\Big(\frac{wb_n}{nk}\Big)$$

for $w = \mathcal{O}(\log^2(n))$. All other computations are identical and we thus omit them. □

Proof of Theorem 4.20. Lemma 4.8 gives us the behaviour of $\mathbb{E}_\Theta[\log \widetilde{Y}_n]$. It is thus enough to compute the expected value of $\widetilde{\Delta}_n = \log \widetilde{Y}_n - \log \widetilde{O}_n$. Equations (4.18) and (4.19) yield

$$\mathbb{E}_\Theta\left[\widetilde{\Delta}_n\right] = \sum_{k=1}^{n} \Lambda(k)\mathbb{E}_\Theta\left[\widetilde{D}_{nk} - \widetilde{D}^*_{nk}\right]. \tag{4.38}$$

Denote $\alpha := \lfloor \log^2(n) \rfloor$ and consider the two sets $S_1 := \{1 \le k \le \alpha\}$ and $S_2 := \{\alpha < k \le n\}$. We split the sum according to the two sets and show first that the second sum is negligible. Indeed, by Proposition 4.5 and (2.27),

$$\begin{aligned}\sum_{k\in S_2} \Lambda(k)\mathbb{E}_\Theta\left[\widetilde{D}_{nk} - \widetilde{D}^*_{nk}\right] &= \mathcal{O}\left(\sum_{k\in S_2} \Lambda(k)\mathbb{E}_\Theta\left[\widetilde{D}_{nk}(\widetilde{D}_{nk}-1)\right]\right)\\ &= \mathcal{O}\left(\sum_{k\in S_2} \Lambda(k)\mathbb{E}_\Theta\left[D_{nk}(D_{nk}-1)\right]\right)\\ &= \mathcal{O}\Bigg(\log^2(n)\sum_{k\in S_2}\frac{\Lambda(k)}{k^2}\Bigg) = \mathcal{O}(1).\end{aligned}$$

Therefore, it is sufficient to consider the sum over the set S_1. Lemma 4.24 then yields for $k \le \log^2(n)$

$$\begin{aligned}&\mathbb{E}_\Theta\left[\widetilde{D}_{nk} - \widetilde{D}^*_{nk}\right] = \mathbb{E}_\Theta\left[\widetilde{D}_{nk}\right] - 1 + \mathbb{P}_\Theta\left[\widetilde{D}^*_{nk} = 0\right]\\ &= \exp\left(-\sum_{m=1}^{b_n}\frac{\theta_m}{m}\rho^m\,\mathbb{1}_{\{k|m\}}\right) - 1 + \sum_{m=1}^{b_n}\frac{\theta_m}{m}\rho^m\,\mathbb{1}_{\{k|m\}} + \mathcal{O}\Big(\frac{b}{nk}\Big).\end{aligned}$$

Since $\Lambda(k) \le \log(k)$, the sum over the error term is of order

$$\frac{b_n}{n}\sum_{k\in S_1}\mathcal{O}\left(\frac{\Lambda(k)}{k}\right) = \mathcal{O}\left(\frac{(\log\log(n))^2}{\log^2(n)}\right) = \mathcal{O}\left(\frac{1}{\log(n)}\right).$$

Altogether, we proved that

$$\mathbb{E}_\Theta\left[\widetilde{\Delta}_n\right] = \sum_{k=1}^{\log^2(n)}\Lambda(k)\left(e^{-\sum_{m=1}^{b_n}\frac{\theta_m}{m}\rho^m\,\mathbb{1}_{\{k|m\}}} - 1 + \sum_{m=1}^{b_n}\frac{\theta_m}{m}\rho^m\,\mathbb{1}_{\{k|m\}}\right) + \mathcal{O}(1).$$

Using the definition of Δ_n and Lemma 4.8 completes the proof. □

4.6 The expected value

We provide in this section a precise expansion of $\mathbb{E}_\Theta[\log O_n]$ which has in particular, as in the truncated setting in the previous section, an interpretation in terms of the Riemann hypothesis. In this section we require additional assumptions on the function g_Θ, namely that $g_\Theta \in \mathcal{LF}(\rho,\vartheta)$, which will be defined in Definition 4.29. For this class of functions we will prove the following

Theorem 4.25. *Suppose that $g_\Theta \in \mathcal{LF}(\rho,\vartheta)$. Then*

$$\begin{aligned}\mathbb{E}_\Theta\left[\log O_n\right] &= \mathbb{E}_\Theta\left[\log Y_n\right] - \vartheta\log(n)\big(1-\log(\vartheta\log(n))\big)\\ &\quad+\sum_{\varrho}\Gamma(-\varrho)\big(\vartheta\log(n)\big)^{\varrho} + \mathcal{O}\left((\log\log(n))^3\right),\end{aligned}\tag{4.39}$$

where $\sum_\varrho$ denotes the sum over the non-trivial zeros of the Riemann zeta function.

This statement together with the Euler summation formula (2.34) yields as immediate consequence

Corollary 4.26. *Suppose that $g_\Theta \in \mathcal{LF}(\rho,\vartheta)$ and that $\theta_m\rho^m = \vartheta + \mathcal{O}(m^{-\delta})$ for some $\delta > 0$. Then*

$$\begin{aligned}\mathbb{E}_\Theta\left[\log O_n\right] &= \frac{\vartheta}{2}\log^2(n) - \vartheta\log(n)\big(1-\log(\vartheta\log(n))\big)\\ &\quad + \sum_{\varrho}\Gamma(-\varrho)(\vartheta\log(n))^{\varrho} + \mathcal{O}\left(\log\log(n)^3\right).\end{aligned}$$

Furthermore, similarly to Corollary 4.22 one obtains

Corollary 4.27. *Suppose that $g_\Theta \in \mathcal{LF}(\rho,\vartheta)$ and that $\theta_m\rho^m = \vartheta + \mathcal{O}(m^{-\delta})$ for some $\delta > 0$. Then following statements are equivalent*

(1) The Riemann hypothesis is true.

(2) We have for all $\epsilon > 0$

$$\mathbb{E}_\Theta\left[\log O_n\right] = \frac{\vartheta}{2}\log^2(n) - \vartheta\log(n)\big(1-\log(\vartheta\log(n))\big) + \mathcal{O}\left(\log(n)^{\frac{1}{2}+\epsilon}\right).$$

Equation (4.39) was proven by Zacharovas in [78] for the uniform measure on $\mathfrak{S}_n$ and in [79] on the subgroup $\mathfrak{S}_n^{(k)} := \{\sigma = \tau^k | \tau \in \mathfrak{S}_n\}$. Zacharovas also noted the implication (1) $\Rightarrow$ (2) of Corollary 4.27, but not the important opposite implication.

Recall that the crucial point in the proof of Theorem 4.20 was the expansion of $\mathbb{E}_\Theta[\widetilde{\Delta}_n]$ as in (4.38) and the expected values of $\widetilde{D}_{nk}$ and $\widetilde{D}^*_{nk}$ for $k \leq \log^2(n)$. We thus start by studying $\mathbb{E}_\Theta\left[D_{nk}\right]$ and $\mathbb{E}_\Theta\left[D^*_{nk}\right]$.

Lemma 4.28. *For $k \in \mathbb{N}$ and $u \in \mathbb{C}$ the following holds:*

(1) $\mathbb{E}_\Theta\left[u^{D_{nk}}\right] = \frac{1}{h_n}t^n[\exp\left(g_\Theta(t) + (u-1)g_{\Theta,k}(t)\right)]$,

(2) $\mathbb{E}_\Theta\left[D_{nk}\right] = \frac{1}{h_n}t^n[g_{\Theta,k}(t)\exp\left(g_\Theta(t)\right)]$,

(3) $\mathbb{P}_\Theta\left[D^*_{nk} = 0\right] = \frac{1}{h_n}t^n[\exp\left(g_\Theta(t) - g_{\Theta,k}(t)\right)]$,

where

$$g_{\Theta,k}(t) = \sum_{m=1}^{\infty}\frac{\theta_m}{m}\mathbb{1}_{\{k|m\}}\,t^m. \tag{4.40}$$

Proof. The proof is very similar to the proof of Lemma 4.23. □

Equation (4.14) implies that $\theta_m \rho^m$ converges to ϑ if $g_\Theta \in \mathcal{F}(\rho, \vartheta, K)$. Thus $g_{\Theta,k}$ has radius of convergence ρ for all k. If we would like to use a similar argument as in Lemma 4.24, we require further assumptions on the function g_Θ. To get a vague intuition, let us have a look at the Ewens measure, meaning that $\theta_m = \vartheta$ for all $m \in \mathbb{N}$. For this model,

$$g_\Theta(t) = \vartheta \log\left(\frac{1}{1 - t/\rho}\right) \quad \text{and} \quad g_{\Theta,k}(t) = \frac{\vartheta}{k} \log\left(\frac{1}{1-(t/\rho)^k}\right).$$

Clearly, each $g_{\Theta,k}(t)$ can be extended beyond its disk of convergence and its singularities are k-th roots of unity. These observations motivate the following definition.

Definition 4.29. *Let $\rho, \vartheta > 0$ be given. We write $\mathcal{LF}(\rho, \vartheta)$ for the set of all functions $g_\Theta(t) = \sum_{m=1}^\infty \frac{\theta_m}{m} t^m$ such that there exists $R > r$ and $0 < \phi < \frac{\pi}{2}$ so that the following conditions are satisfied for all $k \in \mathbb{N}$:*

(1) $g_{\Theta,k}$ defined in (4.40) is holomorphic in a domain

$$\Delta_{0,k}(\rho, R, \phi) := \bigcap_{m=0}^{k-1} e^{\frac{2\pi m i}{k}} \Delta_0(\rho, R, \phi).$$

(2) We have

$$g_{\Theta,k}(t) = \frac{\vartheta}{k} \log\left(\frac{1}{1-(t/\rho)^k}\right) + K_k + \mathcal{O}\left(t - \rho\right) \ \text{as } t \to \rho \tag{4.41}$$

with $\mathcal{O}(\cdot)$ uniform in k and $K_k = \mathcal{O}(1/k)$.

We require for the the proof of Theorem 4.25 the asymptotic behavior of $\mathbb{E}_\Theta\left[D_{nk}\right]$ and $\mathbb{E}_\Theta\left[D_{nk}^*\right]$ for $g_\Theta \in \mathcal{LF}(\rho, \vartheta)$. We have

Lemma 4.30. *Suppose that g_Θ belongs to $\mathcal{LF}(\rho, \vartheta)$, then the following holds uniformly in k for $2 \le k \le n^{\frac{\vartheta}{1+\vartheta}}$:*

(1) $\mathbb{E}_\Theta\left[D_{nk}\right] = \frac{\vartheta}{k} \log\left(\frac{n}{k}\right) + \mathcal{O}\Big(\frac{1}{k} + \frac{k^{\vartheta+1}}{n^\vartheta}\Big)$,

(2) $\mathbb{P}_\Theta\left[D_{nk}^* = 0\right] = \left(\frac{n}{k}\right)^{-\frac{\vartheta}{k}} \frac{\Gamma(\vartheta)}{\Gamma\big(\vartheta(1-\frac{1}{k})\big)} \Big(1 + \mathcal{O}\Big(\frac{1}{n} + \frac{k^{\vartheta+1}}{n^\vartheta}\Big)\Big)$.

Proof. The proof is very similar to the proof of Lemma 4.8. Combine Theorem 4.28 and Cauchy's integral formula (see Theorem 2.11) to obtain

$$h_n \mathbb{E}_\Theta\left[D_{nk}\right] = \frac{1}{2\pi i} \int_\gamma g_{\Theta,k}(t) \exp\left(g_\Theta(t)\right) \frac{dt}{t^{n+1}},$$
$$h_n \mathbb{E}_\Theta\left[D_{nk}^*\right] = \frac{1}{2\pi i} \int_\gamma \exp\left(g_\Theta(t) - g_{\Theta,k}(t)\right) \frac{dt}{t^{n+1}}.$$

By assumption, $g_{\Theta,k}$ is holomorphic in some domain $\Delta_{0,k}(\rho, R, \phi)$ (see Definition 4.29). Following the idea in [39, Section VI.3], we choose the curve γ as in Figure 6, such that γ is contained in $\Delta_{0,k}(\rho, R, \phi)$. More precisely, the radius of the big circle is $R(n) := \rho(1 + b_n)$ with b_n as in (4.17), the radii of the small circles as $1/n$ and the angles of the line segments all equal and independent of n. Let us first show that

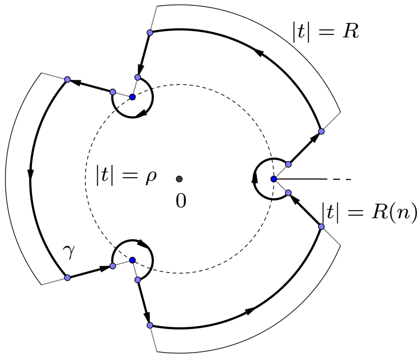


Figure 6: Illustration of the curve γ in proof of Lemma 4.30 with $k = 3$.

the contribution of integral over the big circle is negligible. Since $kb_n = o(n)$, we get

$$|g_{\Theta,k}(t)| \le \vartheta \log \left| \frac{1}{1 - (t/\rho)^k} \right| + \mathcal{O}(1) \implies |g_\Theta(R(n)e^{i\varphi})| \le \vartheta \log(kb_n) + \mathcal{O}(1).$$

The estimates on t^{-n-1} and $g_\Theta(t)$ are the same as in the proof of Lemma 4.8. Combining all three, one immediately realizes that the integral over the outer circle is negligible. It remains to compute the behavior along the curves around the points $\rho \cdot e^{j\frac{2\pi i}{k}}$ for $0 \le j < k$. We have to distinguish the cases $1 \le j < k$ and $j = 0$. For $j = 0$ we are in the same situation as in the proof of Lemma 4.8. With the variable substitution $t = \rho(1 + w/n)$ with $w = \mathcal{O}(\log^2(n))$ the curve around ρ is mapped to the bounded curve γ' in Figure 5(b). Furthermore, on γ' the following expansions hold:

$$\begin{aligned}
g_\Theta(t) &= \vartheta \log(n) - \vartheta \log(-w) + K + \mathcal{O}(w/n),\\
g_{\Theta,k}(t) &= \frac{\vartheta}{k}\big(\log(n/k) - \log(-w)\big) + K_k + \mathcal{O}(w/n),\\
t^{-n-1} &= \rho^{-n-1} e^{-w}(1 + \mathcal{O}(w/n)).
\end{aligned}$$

This implies

$$\begin{aligned}
&\frac{1}{2\pi i} \int_{\gamma_{1,0}\cup\gamma_{2,0}\cup\gamma_{3,0}} \exp\left(g_\Theta(t) - g_{\Theta,k}(t)\right) \frac{dt}{t^{n+1}}\\
=&\frac{n^{\vartheta(1+\frac{u-1}{k})-1} e^{(u-1)K_k}}{\rho^n k^{\vartheta\frac{u-1}{k}} 2\pi i} \int_{\gamma'} (-w)^{-\vartheta(1+\frac{u-1}{k})} e^{-w} (1 + \mathcal{O}(w/n))\, dw.
\end{aligned}$$

As in the proof of Lemma 4.8, one can replace the bounded curve γ' by the Hankel contour γ'' in Figure 5(c). Using again (4.23) and Corollary 3.8 shows that the integral over this part gives the main term in Equation (2) of Lemma 4.30. The argument for (1) is similar.

We now proceed to the case $1 \leq j \leq k-1$ where we need the variable substitution $t = \rho \cdot e^{j\frac{2\pi i}{k}}(1 + w/n)$. The curve $\gamma_j := \gamma_{1,j} \cup \gamma_{2,j} \cup \gamma_{3,j}$ is also mapped to γ', but here the expansions along γ' are given by

$$\begin{aligned}
g_\Theta(t) &= -\vartheta \log\left(1 - e\left(\frac{j}{k}\right)\right) + \mathcal{O}\Big(\frac{kj}{n}\Big),\\
g_{\Theta,k}(t) &= \frac{\vartheta}{k}\big(\log(n/k) - \log(-w)\big) + \mathcal{O}(w/n),\\
t^{-n-1} &= \rho^{-n-1} e\left(\frac{-(n+1)j}{k}\right) e^{-w}(1 + \mathcal{O}(w/n)).
\end{aligned}$$

Insert this into the Cauchy integral and summing over j from 1 to $k-1$ gives the error terms in (1) and (2). $\square$

We are now prepared to prove the main result of this section.

Proof of Theorem 4.25 . The argument is very similar to the one of proof of the Theorem 4.20 and Corollary 4.21. We thus give here only a short overview. Recall that

$$\mathbb{E}_\Theta\left[\log Y_n\right] - \mathbb{E}_\Theta\left[\log O_n\right] = \mathbb{E}_\Theta\left[\Delta_n\right] = \sum_{k=1}^{n} \Lambda(k)\mathbb{E}_\Theta\left[D_{nk} - D^*_{nk}\right].$$

Denote $\alpha := \lfloor \log^2(n) \rfloor$ and consider the two sets $S_1 := \{1 \leq k \leq \alpha\}$ and $S_2 := \{\alpha < k \leq n\}$. As in the proof of Theorem 4.20, we can show that the sum over the second set is negligible. It is thus sufficient to consider only the sum over S_1. Lemma 4.30 yields for $k \leq \log^2(n)$

$$\begin{aligned}
&\mathbb{E}_\Theta\left[D_{nk} - D^*_{nk}\right] = \mathbb{E}_\Theta\left[D_{nk}\right] - 1 + \mathbb{P}_\Theta\left[D^*_{nk} = 0\right]\\
&= \frac{\vartheta}{k}\log(n) - 1 + \Big(\frac{n}{k}\Big)^{-\frac{\vartheta}{k}}\left(1 + \mathcal{O}\left(\frac{1}{k} + \frac{k^{\vartheta+1}}{n^\vartheta}\right)\right) + \mathcal{O}\left(\frac{\log(k)}{k}\right)\\
&= \frac{\vartheta}{k}\log(n) - 1 + e^{-\frac{\vartheta}{k}\log(n)} + \mathcal{O}\left(\frac{\log(k)}{k}\right).
\end{aligned}$$

This is now (almost) the same expression as in the proof of Corollary 4.21. The remaning computations are the same and thus we omit them. $\square$

5

The order of permutations with polynomial cycle weights

In the beginning of Chapter 4 the main results on the order of permutations were presented, namely the asymptotic behavior of the Landau function in (4.1) and the Erdős-Turán law in (4.2). In this chapter, we extend the Erdős-Turán law to random permutations chosen according to the generalized weighted measure $\mathbb{P}_\Theta$ as defined in Definition 1.1 with polynomial cycle weights $\theta_m = m^\gamma$, $\gamma > 0$. One of our motivations is to find weights such that the order of a typical permutation with respect to this measure comes close to the maximum as in Landau's result.

Only a few results are known for these parameters. Ercolani and Ueltschi [30] show that for this model, a typical cycle has length of order $n^{\frac{1}{1+\gamma}}$ and that the total number of cycles has order $n^{\frac{\gamma}{1+\gamma}}$. They also prove that the component process converges in distribution to mutually *independent* Poisson random variables Z_m:

$$(C_1^n, C_2^n, \dots) \xrightarrow{d} (Z_1, Z_2, \dots), \qquad \text{as } n \to \infty. \tag{5.1}$$

For many purposes this convergence is not strong enough, since it only involves the convergence of the vectors $(C_1^n, C_2^n, \dots, C_b^n)$ for fixed b. However, many natural properties of the component process jointly depend on all components, including the large ones, even though their contribution is less relevant. Thus, estimates are needed where b and n grow simultaneously. The quality of the approximation can conveniently be described in terms of the total variation distance. For all $1 \le b \le n$ denote by $\mathrm{d}_b(n)$ the total variation distance

$$\mathrm{d}_b(n) \coloneqq \mathrm{d}_{\mathrm{TV}}\big(\mathcal{L}(C_1^n, C_2^n, \dots, C_b^n), \mathcal{L}(Z_1, Z_2, \dots, Z_b)\big). \tag{5.2}$$

For the uniform measure, where the Z_m are independent Poisson random variables with mean $1/m$, it was proved in 1990 by Barbour [10] that $\mathrm{d}_b(n) \le 2b/n$. This bound may be improved significantly. In 1992 Arratia and Tavaré [9] showed that

$$\mathrm{d}_b(n) \to 0 \quad \text{if and only if} \quad b = o(n). \tag{5.3}$$

In particular, if $b = o(n)$, then $\mathrm{d}_b(n) \to 0$ superexponentially fast with respect to n/b. The extension of these results to the Ewens measure is straightforward (here each Z_m has mean ϑ/m), but superexponential decay of $\mathrm{d}_b(n)$ is only attained for $\vartheta = 1$. For parameters $\vartheta \neq 1$ we have $\mathrm{d}_b(n) = \mathcal{O}(b/n)$; see Arratia et al. [7, Theorem 6]. For the uniform and the Ewens measure, the Feller coupling is used to study $\mathrm{d}_b(n)$. When considering random permutations with respect to the weighted measure $\mathbb{P}_\Theta$, the Feller coupling is not available because of a lack of compatibility between the dimensions. Another approach is needed and it will turn out that for $\theta_m = m^\gamma$ the saddle-point method is the right one to choose. We will prove in Section 5.2 that, for appropriately chosen Poisson random variables Z_m, the following holds:

Theorem 5.1. *Let* $\mathrm{d}_b(n)$ *be defined as in* (5.2) *and assume* $\theta_m = m^\gamma, \gamma > 0$. *Then, as* $n \to \infty$,

$$\mathrm{d}_b(n) \to 0 \quad \textit{if and only if} \quad b = o(n^{\frac{1}{1+\gamma}}). \tag{5.4}$$

Furthermore, if $b = o(n^{\frac{1}{1+\gamma}})$, *then* $\mathrm{d}_b(n) = \mathcal{O}(b^{2+\gamma} n^{-\frac{2+\gamma}{1+\gamma}} + b^{-\frac{\gamma}{6}})$.

For the Ewens measure several applications demonstrating the power of (5.3) are available. Estimates like these unify and simplify proofs of limit theorems for a variety of functionals of the cycle counts, such as a Brownian motion limit theorem for cycle counts and the Erdős-Turán law for the order of a permutation (see (4.2)), among others; see [8] for a detailed account. The basic strategy is as follows: first, choose an appropriate $b = b(n) \to \infty$ and show that the contribution of the cycles of size bigger than b is negligible. Second, approximate the distribution of the cycles of size at most b by the independent limiting process, the error being controlled by the bound on the total variation distance.

Comparing (5.3) and (5.4) we notice that for polynomial parameters, the cycle counts exhibit a more dependent structure. An intuitive interpretation is the following. In the Ewens case, a typical cycle has length of order n, and the numbers of cycles of length $o(n)$ are asymptotically independent of each other. For polynomial parameters $\theta_m = m^\gamma$, a typical cycle has length of order $n^{\frac{1}{1+\gamma}}$ (see [30, Theorem 5.1]), providing an intuitive justification for the bound on b in (5.4).

The condition $b = o(n^{\frac{1}{1+\gamma}})$ in Theorem 5.1 is much more restrictive than the condition $b = o(n)$ for the Ewens measure. Thus, the study of random variables involving almost all cycle counts C_m is more difficult for weights $\theta_m = m^\gamma, \gamma > 0$. The reason is that in many cases the cycles with length longer than $n^{\frac{1}{1+\gamma}}$ have a non-negligible contribution; see also Remark 5.14. However, in Section 5.3 we will show that (5.4) is useful to prove an analog of the Erdős-Turán law (4.2) for our setting. We will prove that for $0 < \gamma < 1$, as $n \to \infty$,

$$\frac{\log O_n - G(n)}{\sqrt{F(n)}} \xrightarrow{d} \mathcal{N}(0,1)$$

holds, where $G(n) = \mathcal{O}\big(n^{\frac{\gamma}{1+\gamma}} \log(n)\big)$ and $F(n) = \mathcal{O}\big(n^{\frac{\gamma}{1+\gamma}} \log^2(n)\big)$; see Theorem 5.11 for the exact statement. In particular, notice that for $0 < \gamma < 1$ there exists constants $c, C > 0$ such that

$$c\, n^{\frac{\gamma}{1+\gamma}} \log(n) \leq \mathbb{E}_\Theta \left[\log O_n\right] \leq C\, n^{\frac{\gamma}{1+\gamma}} \log(n).$$

Thus, for our choice of parameters, the mean of $\log O_n$ is indeed very close to Landau's result (4.1). Unfortunately, our approach does not work for $\gamma \geq 1$ and thus the behavior in this situation is currently unknown.

Furthermore, though the bound in (5.4) is too small to investigate the whole cycle count process via the independent Poisson random variables, we will present in Section 5.4 how (5.4) may be used to study the small components by proving a functional version of the Erdős-Turán law. For $x > 0$ define $x^* := \lfloor x\, n^{\frac{\gamma}{1+\gamma}} \rfloor$ and

$$B_n(x) := \frac{\log O_{x^*} - \frac{1}{1+\gamma}\, x^\gamma \log(n)\, n^{\frac{\gamma^2}{1+\gamma}}}{\frac{\sqrt{\gamma}}{1+\gamma} \log(n)\, n^{\frac{\gamma^2}{2(1+\gamma)}}},$$

where $O_{x^*}(\sigma) := \operatorname{lcm}\{m \leq x^*; C_m > 0\}$. We will prove that for $0 < \gamma < 1$ the process $B_n(x)$ converges weakly to $\mathcal{W}(x^\gamma)$, where $\mathcal{W}$ denotes a standard Brownian motion. Moreover, in Section 5.5 we prove the following precise large deviations estimate:

Theorem 5.2. *Let $\mathbb{P}_\Theta$ be the weighted measure and consider the parameters $\theta_m = m^\gamma$ with $0 < \gamma < 1$. Define*

$$\Omega_n := \frac{\log O_n - \lambda_n \log(n)(1+\gamma)^{-2}}{\lambda_n^{1/3} \log(n)(1+\gamma)^{-2}}$$

where

$$\lambda_n = \frac{(1+\gamma)\Gamma(1+\gamma)}{\Gamma(1+\gamma)^{\frac{\gamma}{1+\gamma}}}\, n^{\frac{\gamma}{1+\gamma}} \mathcal{O}\big(1 + \mathcal{O}(\log^{-1}(n))\big).$$

Then for any $x > 0$ the following asymptotic holds:

$$\mathbb{P}_\Theta \left[\Omega_n \geq x \lambda_n^{1/3}\right] = \frac{\exp(-\lambda_n^{1/3} \frac{x^2}{2} + \frac{x^3}{6})}{x \lambda_n^{1/6} \sqrt{2\pi}}\, (1 + o(1)).$$

5.1 Preliminaries

The asymptotic behavior of all random variables on the group $\mathfrak{S}_n$ with respect to the weighted measure

$$\mathbb{P}_\Theta\left[\sigma\right] := \mathbb{P}_\Theta^n[\sigma] := \frac{1}{h_n n!} \prod_{m=1}^{n} \theta_m^{C_m} \tag{5.5}$$

(see Definition 1.1) strongly depends on the sequence $\Theta = (\theta_m)_{m\geq 1}$. As for the generalized Ewens measure, the starting point of our study is the link of the coefficients $(h_n)_{n\geq 1}$ and the generating series g_Θ which was stated in Corollary 2.4:

$$\sum_{n=0}^{\infty} h_n t^n = \exp(g_\Theta(t)) \quad \text{where} \quad g_\Theta(t) = \sum_{m=1}^{\infty} \frac{\theta_m}{m} t^m. \tag{5.6}$$

We refer the reader to Section 2.1 for an overview of properties of the symmetric group and generating series which we need to establish our results. Recall also the randomized measure $\mathbb{P}_\Theta^t$ which was introduced in Section 2.1. Under this measure, the cycle counts C_m are independent and Poisson distributed with $\mathbb{E}_\Theta^t [C_m] = \frac{\theta_m}{m} t^m$ and the following conditioning relation holds:

$$\mathbb{P}_\Theta^t [\,\cdot \mid \mathfrak{S}_n] = \mathbb{P}_\Theta^n [\,\cdot\,]; \tag{5.7}$$

see (2.6). In Section 5.2 we will compare the distribution of the cycle counts C_m under $\mathbb{P}_\Theta^n$ and under $\mathbb{P}_\Theta^t$. To avoid confusion, we will write Z_m instead of C_m whenever we consider the measure $\mathbb{P}_\Theta^t$. Then the Z_m are independent Poisson random variables with mean $\frac{\theta_m}{m} t^m$. Recall that (5.7) implies the so-called *Conditioning Relation*

$$\mathcal{L}\big((C_1, \ldots, C_n)\big) = \mathcal{L}\Big((Z_1, \ldots, Z_n) \big| \sum_{k=1}^{n} k Z_k = n\Big). \tag{5.8}$$

This important relation is necessary for the proof of Theorem 5.1.

The approximation random variable $\log Y_n$

Recall that the order $O_n(\sigma)$ of a permutation $\sigma \in \mathfrak{S}_n$ is the smallest integer k such that the k-fold application of σ to itself gives the identity. Assume that σ decomposes into disjoint cycles $\sigma_1 \cdots \sigma_\ell$ and denote by λ_i the length of the cycle σ_i. Then the order $O_n(\sigma)$ can be computed as

$$O_n(\sigma) = \operatorname{lcm}(\lambda_1, \lambda_2, \cdots \lambda_\ell).$$

In Section 4.1 we introduced

$$D_{nk} := \sum_{m=1}^{n} C_m \, \mathbb{1}_{\{k|m\}} \quad \text{and} \quad D_{nk}^* := \min\{1, D_{nk}\}$$

and the von Mangoldt function Λ, which is defined as

$$\Lambda(n) = \begin{cases} \log(p) & \text{if } n = p^k \text{ for some prime } p \text{ and } k \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

It was explained that the logarithm of the order can be computed as

$$\log O_n = \sum_{k=1}^{n} \Lambda(k) D_{nk}^* \tag{5.9}$$

and that a good approximation of $\log O_n$ is given by

$$\log Y_n = \sum_{m=1}^{n} \log(m)\, C_m = \sum_{k=1}^{n} \Lambda(k) D_{nk}. \tag{5.10}$$

Define

$$\Delta_n := \log Y_n - \log O_n = \sum_{k \le n} \Lambda(k)\big(D_{nk} - D_{nk}^*\big).$$

Typically, in order to prove properties of $\log O_n$, one first establishes them for $\log Y_n$ and then one needs to show that Δ_n is approximately small enough to transfer the result to $\log O_n$. A good tool to study $\log Y_n$ is its moment generating function. The *randomization method* yields

$$\sum_{n=0}^{\infty} h_n \mathbb{E}_\Theta[\exp(s \log Y_n)] t^n = \exp\left(\sum_{m=1}^{\infty} \frac{\theta_m}{m^{1-s}} t^m\right);$$

see (2.7). For $\theta_m = m^\gamma$ one obtains the generating series

$$\sum_{n=0}^{\infty} h_n \mathbb{E}_\Theta[\exp(s \log Y_n)] t^n = \exp\left(\sum_{m=1}^{\infty} \frac{t^m}{m^{1-s-\gamma}}\right) =: \exp(\hat{g}_\Theta(t,s)). \tag{5.11}$$

As we consider s fixed for the moment, we may write $\hat{g}_\Theta(t)$ instead of $\hat{g}_\Theta(t,s)$. The function $\hat{g}_\Theta(t)$ is known to be the polylogarithm $\mathrm{Li}_a(t)$ with parameter

$$a = 1 - s - \gamma.$$

Its radius of convergence is 1 and as $t \to 1$ it satisfies the following asymptotic for $a \notin \{1, 2, \dots\}$ (see [39, Theorem VI.7]):

$$\mathrm{Li}_a(t) \sim \Gamma(1-a)(-\log(t))^{a-1} + \sum_{j \ge 0} \frac{(-1)^j}{j!} \xi(a-j)(-\log(t))^j. \tag{5.12}$$

In particular, for $a < 1$, (5.12) implies for $t \to 1$

$$\mathrm{Li}_a(t) = \Gamma(1-a)(-\log(t))^{a-1} + \zeta(a) + \mathcal{O}(t-1).$$

We will show that $\hat{g}_\Theta(t,s)$ satisfies the so-called log-admissibility (see Definition 5.3 below) in order to apply the saddle-point method to compute $\mathbb{E}_\Theta[\exp(s \log Y_n)]$.

Saddle-point analysis

The appropriate method to investigate generating functions involving g_Θ depends on the parameters $\Theta = (\theta_m)_{m \ge 1}$. For the choice $\theta_m = m^\gamma$ a suitable method to investigate the behavior of functionals of interest is the saddle-point method. This concept was already presented in Section 2.2; see in particular Definition 2.15 and Theorem 2.16. We define here the so-called *log-admissibility* which can be interpreted as a logarithmic analog of the Hayman-admissibility. In fact, if $g(t)$ is log-admissible, then $G(t) = \exp(g(t))$ is Hayman-admissible.

Definition 5.3 (Log-admissibility). *Let $g(t) = \sum_{n\geq 0} g_n t^n$ be given with radius of convergence $\rho > 0$ and $g_n \geq 0$ for all n. Then $g(t)$ is called* log-*admissible if there exist functions $\alpha, \beta, \delta : [0, \rho) \to \mathbb{R}^+$ and $R : [0, \rho) \times (-\pi/2, \pi/2) \to \mathbb{R}^+$ with the following properties:*

Approximation *For all $|\phi| \leq \delta(r)$ the expansion*

$$g(re^{i\phi}) = g(r) + i\phi\alpha(r) - \frac{\phi^2}{2}\beta(r) + R(r, \phi)$$

holds, where $R(r, \phi) = o(\phi^3\delta(r)^{-3})$.

Divergence *$\alpha(r) \to \infty$, $\beta(r) \to \infty$ and $\delta(r) \to 0$ as $r \to \rho$.*

Width of convergence *We have $\epsilon\delta^2(r)\beta(r) - \log\beta(r) \to \infty$ for all $\epsilon > 0$ as $r \to \rho$.*

Monotonicity *$\mathrm{Re}\big(g(re^{i\phi})\big) \leq \mathrm{Re}\big(g(re^{\pm i\delta(r)})\big)$ holds for all $|\phi| > \delta(r)$.*

In order to study the moment generating function of $\log Y_n$ we will need to study functions g with an additional dependence on a parameter s; see (5.11). Thus, consider the generating series

$$\exp(g(t, s)) = \sum_{n=0}^{\infty} G_{n,s} t^n.$$

The coefficients of Hayman-admissible function can be systematically analyzed with Theorem 2.16. In our setting, the following lemma applies to log-admissible functions.

Lemma 5.4. *Let $I \subset \mathbb{R}$ be interval and suppose that $g(t, s)$ is a smooth function for $s \in I$ and $|t| \leq \rho$. Suppose further that $g(t, s)$ is* log-*admissible in t for all $s \in I$ with associated functions α_s, β_s. Let further r_{xs} be the unique solution of $\alpha_s(r) = x$. If the requirements of Definition 5.3 are fulfilled uniformly in s for s bounded, then, as $n \to \infty$, the following asymptotic expansion holds:*

$$G_{n,s} = \frac{1}{\sqrt{2\pi}}(r_{ns})^{-n}\beta_s(r_{ns})^{-1/2}\exp(g(r_{ns}, s))\big(1 + o(1)\big)$$

uniformly in s for s bounded.

The proof of Lemma 5.4 is analogous to the proof of Proposition 2.2 in [57]; one simply has to verify that all involved expression are uniform in s. This is straightforward and we thus omit the details.

Remark 5.5. It is often difficult to find the exact solution of the saddle-point equation (2.11), that is $\alpha_s(r) = n$. However, it is enough to find r_{ns} with

$$\alpha_s(r_{ns}) = n + o\left(\sqrt{\beta_s(r_{ns})}\right), \tag{5.13}$$

since then the contribution of the error term is negligible in the limit.

Let us apply this method to investigate the asymptotic behavior of h_n given in (5.5). Notice that (5.6) implies

$$h_n = [t^n] \exp(g_\Theta(t)).$$

We have to show that g_Θ is log-admissible. This will be proved in a more general way in Lemma 5.12 in Section 5.3. Then Lemma 5.4 yields

Corollary 5.6. *Let g_Θ be as in* (5.6) *with $\theta_m = m^\gamma$, $\gamma > 0$. Then*

$$\begin{aligned} h_n = \big(2\pi\Gamma(2+\gamma)\big)^{-\frac{1}{2}} \Big(\frac{\Gamma(1+\gamma)}{n}\Big)^{\frac{2+\gamma}{2(1+\gamma)}} \times \\ \exp\left(\frac{1+\gamma}{\gamma}\,\Gamma(1+\gamma)^{\frac{1}{1+\gamma}}\, n^{\frac{\gamma}{1+\gamma}} + \zeta(1-\gamma)\right)\big(1+o(1)\big). \end{aligned}$$

Proof. This is a special case of the proof of Proposition 5.13 in Section 5.3. □

Remark 5.7. We will need for the proof of the rate of convergence in Theorem 5.1 a more precise asymptotic expansion for $G_{n,s}$ than the one in Lemma 5.4. This can be obtained by taking into account more error terms in the $\phi-$expansion of $g(t,s)$ at $t = r$. Often one can indeed obtain a complete asymptotic expansion. The details are explained for instance in [39, Chapter VIII]. For us this means that if

$$R(r,\phi) = c_n(r)\phi^3 + \mathcal{O}\left(d_n(r)\phi^4\right)$$

then the $o(1)$ error-term in Theorem 5.1 is

$$\mathcal{O}\left(\frac{d_n(r_{ns})}{\beta_s(r_{ns})^2} + \frac{c_n(r_{ns})^2}{\beta_s(r_{ns})^3}\right).$$

Applying this to h_n gives

$$\begin{aligned} h_n = \big(2\pi\Gamma(2+\gamma)\big)^{-\frac{1}{2}} \Big(\frac{\Gamma(1+\gamma)}{n}\Big)^{\frac{2+\gamma}{2(1+\gamma)}} \times \\ \exp\left(\frac{1+\gamma}{\gamma}\,\Gamma(1+\gamma)^{\frac{1}{1+\gamma}}\, n^{\frac{\gamma}{1+\gamma}} + \zeta(1-\gamma)\right)\big(1+\mathcal{O}(n^{-\frac{\gamma}{1+\gamma}})\big). \end{aligned}$$

5.2 Total variation distance

This section is devoted to the proof of Theorem 5.1. Recall that we denote by $Z_1, Z_2, \dots$ independent Poisson random variables (their mean will be determinded below). Define

$$T_{\ell k} := \sum_{m=\ell+1}^{k} m\, Z_m.$$

and recall also that the Conditioning Relation (5.8) holds. We rewrite it as

$$\mathcal{L}\big((C_1^n, C_2^n, \ldots, C_n^n)\big) = \mathcal{L}\big((Z_1, Z_2, \ldots, Z_n) | T_{0n} = n\big). \tag{5.14}$$

Recall that we denote by $\mathrm{d}_b(n)$ the total variation distance

$$\mathrm{d}_b(n) = \mathrm{d}_{\mathrm{TV}}\left(\mathcal{L}(C_1^n, C_2^n, \ldots, C_b^n), \mathcal{L}(Z_1, Z_2, \ldots, Z_b)\right). \tag{5.15}$$

We have to prove that

$$\mathrm{d}_b(n) \to 0 \quad \text{ if and only if } \quad b = o(n^{\frac{1}{1+\gamma}}).$$

Given the Conditioning Relation (5.14), Lemma 1 in [8] gives a formula that reduces the total variation distance of two vectors to the distance of two one-dimensional random variables:

$$\mathrm{d}_b(n) = \mathrm{d}_{\mathrm{TV}}\left(\mathcal{L}(T_{0b}), \mathcal{L}(T_{0b} | T_{0n} = n)\right). \tag{5.16}$$

Then $\mathrm{d}_b(n) \to 0$ implies that conditioning on the event $\{T_{0n} = n\}$ does not change the distribution of T_{0b} very much, which is indeed the case when $\{T_{0n} = n\}$ is relatively likely. Recall that for uniform random permutations (5.3) holds; in this setting, one can compute that $\mathbb{P}_\Theta^t\left[T_{0n} = n\right]$ is approximately n^{-1} for n large enough. For polynomial cycle weights, $\mathbb{P}_\Theta^t\left[T_{0n} = n\right]$ is approximately $n^{-1+\frac{\gamma}{2(1+\gamma)}}$ for n large enough, which means that the event $\{T_{0n} = n\}$ is even more likely. Thus, at a first glance it seems promising to compare the distributions of $(C_1^n, C_2^n, \ldots, C_b^n)$ and $(Z_1, Z_2, \ldots, Z_b)$.

When $\mathfrak{S}_n$ is equipped with the uniform or Ewens measure, not only the Conditioning Relation (5.14) holds, but additionally the approximating random variables Z_m satisfy the so-called *Logarithmic Condition*

$$m\,\mathbb{E}[Z_m] \to \vartheta, \qquad \text{as } m \to \infty. \tag{5.17}$$

A variety of well known combinatorial objects which decompose into elementary components (permutations decompose into cycles, graphs into connected components, polynomials into irreducible factors) satisfy the Conditioning Relation and the Logarithmic Condition (see [4, Chapter 2] for a comprehensive overview of examples of logarithmic and non-logarithmic combinatorial structures). For this class of objects, Arratia et al. [6] developed a unified approach to study the total variation distance (5.15) only using the Conditioning Relation and the Logarithmic Condition. By the independence of the random variables Z_m, Arratia and Tavaré [8] rewrite the right-hand side of (5.16) as

$$\begin{aligned}\mathrm{d}_b(n) &= \sum_{k \geq 0} \left(\mathbb{P}_\Theta^t\left[T_{0b} = k\right] - \mathbb{P}_\Theta^t\left[T_{0b} = k | T_{0n} = n\right]\right)^+ \\ &= \sum_{k \geq 0} \mathbb{P}_\Theta^t\left[T_{0b} = k\right] \left(1 - \frac{\mathbb{P}_\Theta^t\left[T_{bn} = n-k\right]}{\mathbb{P}_\Theta^t\left[T_{0n} = n\right]}\right)^+. \end{aligned} \tag{5.18}$$

The key to the the analysis of the accuracy of the approximation is some local limit approximation of the distribution of $T_{bn} = \sum_{m=b+1}^{n} m\, Z_m$. In [6] it is shown that the Logarithmic Condition ensures that $n^{-1}T_{bn} \to X_\vartheta$ in distribution, where X_ϑ is a random variable only depending on ϑ and $b = o(n)$. Via this limiting behavior they establish

$$k\mathbb{P}[T_{bn} = k] \sim \vartheta\, \mathbb{P}[k - n \le T_{bn} \le k - b],$$

which provides the required local limit approximation. Then their main result ([6, Theorem 3.1]) is that for all combinatorial structures satisfying (5.14) and (5.17), considered with respect to the Ewens measure, the following holds:

$$\mathrm{d}_b(n) = \mathrm{d}_{\mathrm{TV}}\left(\mathcal{L}(C_1^n, C_2^n, \ldots, C_b^n), \mathcal{L}(Z_1, Z_2, \ldots, Z_b)\right) \to 0 \quad \text{ for } b = o(n).$$

In this section we consider random permutations with respect to a weighted measure with parameters $\theta_m = m^\gamma$. As mentioned before, the Feller coupling is not available in this situation. Recall Remark 2.6 and the estimate for h_n given in Corollary 5.6. This implies that the convergence in (5.1) holds, where the Z_m are independent Poisson random variables with mean

$$\mathbb{E}_\Theta^t\left[Z_m\right] = \frac{\theta_m}{m} t^m = m^{\gamma-1} t^m,$$

and

$$t = \exp(-\eta_\gamma) \quad \text{ with } \quad \eta_\gamma = \left(\frac{n}{\Gamma(1+\gamma)}\right)^{-\frac{1}{1+\gamma}}. \tag{5.19}$$

Unfortunately, the Logarithmic Condition (5.17) is clearly not satisfied, and thus a different approach is needed to prove Theorem 5.1. The starting point is equation (5.18). We will show that T_{0b}, properly rescaled, can be approximated by a Gaussian random variable G_{0b} with appropriately chosen mean and variance. This enables us to prove that the sum $\sum \mathbb{P}_\Theta^t\left[T_{0b} = k\right]$ converges to zero outside a small interval around the mean of T_{0b}. Within this interval, we will show that the quotient $\mathbb{P}_\Theta^t\left[T_{bn} = n - k\right] / \mathbb{P}_\Theta^t\left[T_{0n} = n\right]$ converges to 1. Let us first compute

$$\mu_{0b} := \mathbb{E}_\Theta^t\left[T_{0b}\right], \; \mu_{bn} := \mathbb{E}_\Theta^t\left[T_{bn}\right], \; \sigma_{0b}^2 := \mathbb{V}_\Theta^t\left[T_{0b}\right] \; \text{ and } \; \sigma_{bn}^2 := \mathbb{V}_\Theta^t\left[T_{bn}\right].$$

Lemma 5.8. *Recall that Σ_2 is defined in (2.29). For $b = o(n^{\frac{1}{1+\gamma}})$ we have*

(1) $\mu_{0b} = \dfrac{1}{1+\gamma} b^{1+\gamma} - \dfrac{n}{\Gamma(1+\gamma)} \Sigma_2(1+\gamma, b\eta_\gamma) + \mathcal{O}(b^\gamma),$

(2) $\sigma_{0b}^2 = \dfrac{1}{2+\gamma} b^{2+\gamma} - \left(\dfrac{n}{\Gamma(1+\gamma)}\right)^{\frac{2+\gamma}{1+\gamma}} \Sigma_2(2+\gamma, b\eta_\gamma) + \mathcal{O}(b^{1+\gamma}),$

(3) $\mu_{bn} = n - \dfrac{1}{1+\gamma} b^{1+\gamma} + \dfrac{n}{\Gamma(1+\gamma)} \Sigma_2(1+\gamma, b\eta_\gamma) + \mathcal{O}(n^{\frac{\gamma}{1+\gamma}}),$

(4) $\sigma_{bn}^2 = \frac{1+\gamma}{\Gamma(1+\gamma)^{\frac{1}{1+\gamma}}} n^{\frac{2+\gamma}{1+\gamma}} - \frac{1}{2+\gamma} b^{2+\gamma} + \left(\frac{n}{\Gamma(1+\gamma)}\right)^{\frac{2+\gamma}{1+\gamma}} \Sigma_2(2+\gamma, b\eta_\gamma) + \mathcal{O}(n).$

Proof. Recall (2.28), (2.30) and (2.33). Then $\mu_{0b} = \mathbb{E}_\Theta^t [T_{0b}]$ is given by

$$\mu_{0b} = \sum_{k=1}^{b} \theta_k t^k = \sum_{k=1}^{b} k^\gamma t^k$$

and (2.33) yields

$$\begin{aligned}\sum_{k=1}^{b} k^\gamma t^k = \int_1^b x^\gamma t^x dx + \gamma \int_1^b (x - \lfloor x \rfloor) x^{\gamma-1} t^x dx \\ + \log(t) \int_1^b (x - \lfloor x \rfloor) x^\gamma t^x dx + b^\gamma t^b (b - \lfloor b \rfloor).\end{aligned}$$

For the first integral, set $t = \exp(-\eta_\gamma)$ with η_γ as in (5.19). With a variable substitution $y = x\eta_\gamma$, we obtain

$$\begin{aligned}\int_1^b x^\gamma e^{-x\eta_\gamma} dx &= \frac{n}{\Gamma(1+\gamma)} \int_{\eta_\gamma}^{b\eta_\gamma} y^\gamma e^{-y} dy \\ &= \frac{n}{\Gamma(1+\gamma)} \big(\Gamma(1+\gamma, \eta_\gamma) - \Gamma(1+\gamma, b\eta_\gamma)\big) \\ &= \frac{1}{1+\gamma} b^{1+\gamma} - \frac{n}{\Gamma(1+\gamma)} \Sigma_2(1+\gamma, b\eta_\gamma) + \mathcal{O}(1),\end{aligned}$$

where the last step follows from (2.28) and $b = o(n^{\frac{1}{1+\gamma}})$. For the remaining terms one can show that they are of order $\mathcal{O}(b^\gamma)$, which yields assertion (1). Similarly,

$$\sigma_{0b}^2 = \sum_{k=1}^{b} k\theta_k t^k = \sum_{k=1}^{b} k^{1+\gamma} t^k$$

and by (2.33)

$$\begin{aligned}\sigma_{0b}^2 &= \int_1^b x^{1+\gamma} e^{-x\eta_\gamma} dx + \mathcal{O}(b^{1+\gamma}) \\ &= \left(\frac{n}{\Gamma(1+\gamma)}\right)^{\frac{2+\gamma}{1+\gamma}} \big(\Gamma(2+\gamma, \eta_\gamma) - \Gamma(2+\gamma, b\eta_\gamma)\big) + \mathcal{O}(b^{1+\gamma}) \\ &= \frac{1}{2+\gamma} b^{2+\gamma} - \left(\frac{n}{\Gamma(1+\gamma)}\right)^{\frac{2+\gamma}{1+\gamma}} \Sigma_2(2+\gamma, b\eta_\gamma) + \mathcal{O}(b^{1+\gamma}),\end{aligned}$$

proving (2). The computations for T_{bn} are analogous. In particular, notice that

$$\mu_{0b} + \mu_{bn} = \mu_{0n} = n + \mathcal{O}(1). \tag{5.20}$$

The proof is complete. $\square$

For two real random variables X and Y recall the definition of the Kolmogorov distance $\mathrm{d_K}(X,Y)$ in (2.20). Now define for $x = 1 + \frac{\gamma}{3}$

$$T_{0b}^{x} := \frac{T_{0b}}{b^x}, \quad \mu_{0b^x} := \frac{\mu_{0b}}{b^x} \quad \text{and} \quad \sigma_{0b^x} := \frac{\sigma_{0b}}{b^x}. \tag{5.21}$$

Lemma 5.9. *Assume $b = o(n^{\frac{1}{1+\gamma}})$ let G_{0b} be a Gaussian random variable with mean μ_{0b^x} and variance σ_{0b^x}. Then*

$$\mathrm{d_K}(T_{0b}^{x}, G_{0b}) = \mathcal{O}\big(\sigma_{0b^x}^{-1}\big) = \mathcal{O}\big(b^{-\gamma/6}\big). \tag{5.22}$$

Proof. We will show that T_{0b}^{x} is mod-Gaussian convergent with parameters μ_{0b^x} and σ_{0b^x} (see Definition 2.17 for the definition of this type of convergence). Then the assertion of the lemma is a direct consequence of Proposition 2.18.

The characteristic function of T_{0b} is given by

$$\begin{aligned}
\mathbb{E}_{\Theta}^{t}[e^{isT_{0b}}] &= \exp\Big(\sum_{k=1}^{b} \frac{\theta_k}{k} t^k (e^{isk} - 1)\Big) \\
&= \exp\Big(is\sum_{k=1}^{b} \theta_k t^k - \frac{s^2}{2}\sum_{k=1}^{b} k\theta_k t^k - \frac{is^3}{6}\sum_{k=1}^{b} k^2\theta_k t^k + \mathcal{O}\big(s^4 \sum_{k=1}^{b} k^3\theta_k t^k\big)\Big)
\end{aligned}$$

and we need to find an appropriate scaling such that the third term converges to a constant and the error term converges to zero. In Lemma 5.8 we have given

$$\mu_{0b} = \sum_{k=1}^{b} \theta_k t^k \quad \text{and} \quad \sigma_{0b}^2 = \sum_{k=1}^{b} k\theta_k t^k.$$

Similarly, we compute

$$\begin{aligned}
\sum_{k=1}^{b} k^2\theta_k t^k &= \int_1^b x^{\gamma+2} e^{-x\eta_\gamma} dx + \mathcal{O}(b^{2+\gamma}) \\
&= \Big(\frac{n}{\Gamma(1+\gamma)}\Big)^{\frac{3+\gamma}{1+\gamma}} \big(\Gamma(3+\gamma, \eta_\gamma) - \Gamma(3+\gamma, b\eta_\gamma)\big) \\
&= \frac{1}{3+\gamma} b^{3+\gamma} - \Big(\frac{n}{\Gamma(1+\gamma)}\Big)^{\frac{3+\gamma}{1+\gamma}} \Sigma_2(3+\gamma, b\eta_\gamma) + \mathcal{O}(1)
\end{aligned}$$

and

$$\sum_{k=1}^{b} k^3\theta_k t^k = \mathcal{O}(b^{4+\gamma}).$$

We therefore have to rescale by $s^x = s/b^x$ such that $b^{3+\gamma-3x}$ converges to a constant. Thus, choose $x = 1 + \frac{\gamma}{3}$, then for $T_{0b}^{x} := T_{0b}/b^x$ we get

$$\mathbb{E}_{\Theta}^{t}[e^{isT_{0b}^{x}}] = \exp\Big(is\mu_{0b^x} - \frac{s^2}{2}\sigma_{0b^x}^2 - \frac{is^3}{6}\delta_{0b^x} + \mathcal{O}\big(s^4 b^{-\frac{\gamma}{3}}\big)\Big)$$

where $\mu_{0b^x} = \mu_{0b}/b^x$, $\sigma^2_{0b^x} = \sigma^2_{0b}/b^{2x}$ and

$$\delta_{0b^x} = b^{-3(1+\frac{\gamma}{3})} \sum_{k=1}^{b} k^{\gamma+2} t^k = \frac{1}{3+\gamma} + \mathcal{O}(bn^{-\frac{1}{1+\gamma}}).$$

This completes the proof. □

With these preliminary results at hand, we are prepared to prove Theorem 5.1.

Proof of Theorem 5.1. Assume first $b = o(n^{\frac{1}{1+\gamma}})$ and recall equation (5.18). Since the $(..)^+$-term in (5.18) satisfies $(..)^+ \leq 1$, we want to find α, β such that both sums

$$\sum_{k=0}^{\alpha} \mathbb{P}^t_\Theta[T_{0b} = k] \quad \text{and} \quad \sum_{k=\beta}^{\infty} \mathbb{P}^t_\Theta[T_{0b} = k]$$

converge to zero. Recall the definition of T^x_{0b}, μ_{0b^x} and σ_{0b^x} in (5.21). As in Lemma 5.9, denote by G_{0b} a Gaussian random variable with mean μ_{0b^x} and standard deviation σ_{0b^x}. Let g be any function with $g(b) \to \infty$ and define

$$\epsilon_b \coloneqq \sigma_{0b}\, g(b).$$

Then, as $n \to \infty$, we have

$$\mathbb{P}^t_\Theta\big[\mu_{0b} - \epsilon_b \leq T_{0b} \leq \mu_{0b} + \epsilon_b\big] \to 1.$$

To see this, notice that for $\epsilon^x_b \coloneqq \epsilon_b/b^x$,

$$\begin{aligned} &\mathbb{P}^t_\Theta\big[\mu_{0b} - \epsilon_b \leq T_{0b} \leq \mu_{0b} + \epsilon_b\big] \\ =&\mathbb{P}^t_\Theta\big[\mu_{0b^x} - \epsilon^x_b \leq T^x_{0b} \leq \mu_{0b^x} + \epsilon^x_b\big] \\ =&\mathbb{P}^t_\Theta\big[\mu_{0b^x} - \epsilon^x_b \leq G_{0b} \leq \mu_{0b^x} + \epsilon^x_b\big] + \mathcal{O}\big(\,\mathrm{d_K}(T^x_{0b}, G_b)\big). \end{aligned}$$

Now Lemma 5.9 yields $\mathrm{d_K}(T^x_{0b}, G_b) = \mathcal{O}\big(b^{-\gamma/6}\big)$. By basic properties of the Gaussian distribution,

$$\begin{aligned} \mathbb{P}^t_\Theta\big[\mu_{0b^x} - \epsilon^x_b \leq G_{0b} \leq \mu_{0b^x} + \epsilon^x_b\big] &= \frac{1}{2}\Big(\operatorname{erf}\Big(\frac{\epsilon^x_b}{\sqrt{2}\sigma_{0b^x}}\Big) - \operatorname{erf}\Big(-\frac{\epsilon^x_b}{\sqrt{2}\sigma_{0b^x}}\Big)\Big) \\ &= \frac{1}{2}\Big(\operatorname{erf}\Big(\frac{g(b)}{\sqrt{2}}\Big) - \operatorname{erf}\Big(-\frac{g(b)}{\sqrt{2}}\Big)\Big), \end{aligned}$$

where $\operatorname{erf}(x)$ denotes the error function, which satisfies the asymptotics (2.31) and (2.32). Thus, as $n \to \infty$, for all g with $g(b) \to \infty$,

$$\mathbb{P}^t_\Theta\big[\mu_{0b} - \epsilon_b \leq T_{0b} \leq \mu_{0b} + \epsilon_b\big] = 1 + \mathcal{O}\big(g^{-1}(b)e^{-g^2(b)} + b^{-\gamma/6}\big) \tag{5.23}$$

and therefore both sums

$$\sum_{k=0}^{\mu_{0b}-\epsilon_b} \mathbb{P}^t_\Theta[T_{0b} = k] \quad \text{and} \quad \sum_{k=\mu_{0b}+\epsilon_b}^{\infty} \mathbb{P}^t_\Theta[T_{0b} = k]$$

are of order $\mathcal{O}\big(g^{-1}(b)e^{-g^2(b)} + b^{-\gamma/6}\big)$. Next, in view of (5.18), we have to show that the sum

$$\sum_{k=\mu_{0b}-g(b)\,\sigma_{0b}}^{\mu_{0b}+g(b)\,\sigma_{0b}} \mathbb{P}_{\Theta}^{t}[T_{0b}=k]\left(1-\frac{\mathbb{P}_{\Theta}^{t}\left[T_{bn}=n-k\right]}{\mathbb{P}_{\Theta}^{t}\left[T_{0n}=n\right]}\right)^{+} \tag{5.24}$$

converges to zero. Recall (5.20) : $\mu_{bn}=n-\mu_{0b}$, and denote $I_b := [-g(b)\,\sigma_{0b}, g(b)\,\sigma_{0b}]$. Then we can rewrite the previous sum as

$$\begin{aligned}(5.24) &= \sum_{j\in I_b}\mathbb{P}_{\Theta}^{t}[T_{0b}=\mu_{0b}-j]\left(1-\frac{\mathbb{P}_{\Theta}^{t}\left[T_{bn}=\mu_{bn}+j\right]}{\mathbb{P}_{\Theta}^{t}\left[T_{0n}=n\right]}\right)^{+}\\ &\le \sup_{j\in I_b}\left(1-\frac{\mathbb{P}_{\Theta}^{t}\left[T_{bn}=\mu_{bn}+j\right]}{\mathbb{P}_{\Theta}^{t}\left[T_{0n}=n\right]}\right)^{+} \end{aligned}\tag{5.25}$$

and we have to show that this term converges to 0.

Let us first give an heuristic argument why this should be true. First, one can show that

$$\frac{\mathbb{P}_{\Theta}^{t}\left[T_{bn}=\mu_{bn}+j\right]}{\mathbb{P}_{\Theta}^{t}\left[T_{0n}=n\right]}\to 1 \quad \text{if and only if} \quad \frac{\mathbb{P}_{\Theta}^{t}\left[T_{bn}=\mu_{bn}+j\right]}{\mathbb{P}_{\Theta}^{t}\left[T_{bn}=\mu_{bn}\right]}\to 1.$$

Similarly to (5.21) and Lemma 5.9, we can show that $T_{bn}^{y} := T_{bn}/n^{y}$ with $y=\frac{3+\gamma}{3(1+\gamma)}$ is approximately Gaussian with mean $\mu_{bn}^{y} := \mu_{bn}/n^{y}$ and standard deviation $\sigma_{bn}^{y} := \sigma_{bn}/n^{y}$. Thus, vaguely, let us consider for a moment that T_{bn} is approximately (a discrete version of a) Gaussian random variable G_{bn} with mean μ_{bn} and variance σ_{bn}^2. Then, for $\delta = o(j)$, the question is for which j the following holds:

$$\mathbb{P}_{\Theta}^{t}\left[\mu_{bn}+j\le G_{bn}\le \mu_{bn}+j+\delta\right]\sim \mathbb{P}_{\Theta}^{t}\left[\mu_{bn}\le G_{bn}\le \mu_{bn}+\delta\right].$$

By the standard properties of the Gaussian distribution, this holds for any $j = o(\sigma_{bn})$. Thus, the crucial point why (5.25) should converge to zero is that

$$|j|\le g(b)\,\sigma_{0b}=o(\sigma_{bn}). \tag{5.26}$$

We have $\sigma_{0b}=o(\sigma_{bn})$ and since $g(b)\to\infty$ may be chosen arbitrarily this implies $g(b)\,\sigma_{0b}=o(\sigma_{bn})$:

$$\frac{g(b)\,\sigma_{0b}}{\sigma_{bn}}=\mathcal{O}\big(b^{\frac{2+\gamma}{2}}n^{-\frac{2+\gamma}{2(1+\gamma)}}g(b)\big)$$

and now choose

$$g(b) := \big(n^{\frac{1}{1+\gamma}}b^{-1}\big)^{\frac{\gamma}{2}} \tag{5.27}$$

to get

$$\frac{g(b)\,\sigma_{0b}}{\sigma_{bn}}=\mathcal{O}\big(bn^{-\frac{1}{1+\gamma}}\big)\to 0.$$

For the rigorous proof that (5.25) converges to 0, we compute $\mathbb{P}_\Theta^t\left[T_{bn}=\mu_{bn}+j\right]$ explicitly by means of saddle-point analysis. We have,

$$\mathbb{E}_\Theta^t[u^{T_{bn}}]=\prod_{k=b+1}^{n}\mathbb{E}_\Theta^t\left[u^{kZ_k}\right]=\exp\Big(\sum_{k=b+1}^{n}\frac{\theta_k}{k}t^k(u^k-1)\Big),$$

where t is as in (5.19). Now, for $m\le n$,

$$\mathbb{P}_\Theta^t[T_{bn}=m]=e^{-S_b(t)}t^m[u^m]\exp\Big(\sum_{k=b+1}^{n}k^{\gamma-1}u^k\Big)=e^{-S_b(t)}t^m[u^m]\exp\Big(\sum_{k=b+1}^{\infty}k^{\gamma-1}u^k\Big),$$

where $S_b(t):=\sum_{k=b+1}^{n}k^{\gamma-1}t^k$. Notice that $\mu_{bn}+j\le n$ for large n with the above chosen $g(n)$. In particular, for $b=0$ and $m=n$, we get

$$\mathbb{P}_\Theta^t[T_{0n}=n]=e^{-S_0(t)}t^n[u^n]\exp\big(g_\Theta(u)\big)=e^{-S_0(t)}t^nh_n,$$

where h_n is as in Corollary 5.6. For $b\ne 0$, to prove that $g_{\Theta,b}(u):=\sum_{k=b+1}^{\infty}k^{\gamma-1}u^k$ is log-admissible one proceeds along the same lines as in the proof of Lemma 5.12. The leading term of the saddle-point solution

$$\alpha(r_m)=m+o(\sqrt{\beta(r_m)})$$

is given by $r_m=\exp(-v_m)$ with

$$v_m=\Big(\frac{m+\mu_{0b}}{\Gamma(1+\gamma)}\Big)^{-\frac{1}{1+\gamma}}.$$

Lemma 5.4 together with Remark 5.7 yields

$$[u^m]\exp\Big(\sum_{k=b+1}^{\infty}k^{\gamma-1}u^k\Big)=\frac{1}{\sqrt{2\pi\beta(r_m)}}\exp\big(g_{\Theta,b}(r_m)+m\,v_m\big)\big(1+\mathcal{O}(n^{-\frac{\gamma}{1+\gamma}})\big).$$

Thus we have

$$\frac{\mathbb{P}_\Theta^t[T_{bn}=m]}{\mathbb{P}_\Theta^t[T_{0n}=n]}=\frac{t^{m-n}}{h_n\sqrt{2\pi\beta(r_m)}}\exp\Big(g_{\Theta,b}(r_m)+m\,v_m+\sum_{k=1}^{b}k^{\gamma-1}t^k\Big)\big(1+\mathcal{O}(n^{-\frac{\gamma}{1+\gamma}})\big).$$

Recall $t=\exp(-\eta_\gamma)$ with $\eta_\gamma=\big(\frac{n}{\Gamma(1+\gamma)}\big)^{-\frac{1}{1+\gamma}}$. Let us first compute $\beta(r_m)$. Similarly as in the proof of Lemma 5.8, we have

$$\begin{aligned}\beta(r_m)&=\sum_{k=b+1}^{\infty}k^{1+\gamma}r_m^k=\sum_{k=1}^{\infty}k^{1+\gamma}r_m^k-\sum_{k=1}^{b}k^{1+\gamma}r_m^k\\&=\mathrm{Li}_{-\gamma-1}(r_m)-\int_1^b x^{1+\gamma}e^{-xv_{\tilde m}}dx+\mathcal{O}(b^{1+\gamma})\\&=\Gamma(2+\gamma)v_m^{-(2+\gamma)}+\mathcal{O}\big(b^{2+\gamma}\big).\end{aligned}$$

Together with h_n as in Corollary 5.6 we get

$$\frac{\mathbb{P}_\Theta^t[T_{bn} = m]}{\mathbb{P}_\Theta^t[T_{0n} = n]} = H_{\Theta,b}(r_m, t) \exp\big(G_{\Theta,b}(r_m, t)\big)\big(1 + \mathcal{O}(n^{-\frac{\gamma}{1+\gamma}})\big),$$

with

$$H_{\Theta,b}(r_m, t) = \big(v_m^{-(2+\gamma)} + \mathcal{O}(b^{2+\gamma})\big)^{-1/2}\Big(\frac{n}{\Gamma(1+\gamma)}\Big)^{\frac{2+\gamma}{2(1+\gamma)}}$$

and

$$G_{\Theta,b}(r_m, t) = g_{\Theta,b}(r_m) + m\,(v_m - \eta_\gamma) + \sum_{k=1}^{b} k^{\gamma-1}t^k - \frac{n\,\eta_\gamma}{\gamma} - \zeta(1-\gamma).$$

Recall that we are interested in $\widetilde{m} \coloneqq \mu_{bn} + j$ with $j \in I_b$ so that

$$v_{\widetilde{m}} = \Big(\frac{n+j}{\Gamma(1+\gamma)}\Big)^{-\frac{1}{1+\gamma}} = \eta_\gamma - \frac{\Gamma(1+\gamma)^{\frac{1}{1+\gamma}}}{1+\gamma} j n^{-\frac{2+\gamma}{1+\gamma}} + \mathcal{O}\big(j^2 n^{-\frac{3+2\gamma}{1+\gamma}}\big).$$

Thus,

$$\begin{aligned} H_{\Theta,b}(r_{\widetilde{m}}, t) &= \bigg(\Big(\frac{n+j}{\Gamma(1+\gamma)}\Big)^{-\frac{2+\gamma}{2(1+\gamma)}} + \mathcal{O}\big(n^{-\frac{3(2+\gamma)}{2(1+\gamma)}} b^{2+\gamma}\big)\bigg)\Big(\frac{n}{\Gamma(1+\gamma)}\Big)^{\frac{2+\gamma}{2(1+\gamma)}} \\ &= 1 + \mathcal{O}\Big(j n^{-1} + n^{-\frac{2(2+\gamma)}{2(1+\gamma)}} b^{2+\gamma}\Big), \end{aligned}$$

and the error term converges to zero since for $g(b)$ as in (5.27) we have $|j| \le g(b)\,\sigma_{0b} = o(n)$. It remains to compute $G_{\Theta,b}(r_{\widetilde{m}}, t)$. First notice that

$$\widetilde{m}(v_{\widetilde{m}} - \eta_\gamma) = -\mu_{bn}\frac{\Gamma(1+\gamma)^{\frac{1}{1+\gamma}}}{1+\gamma} j n^{-\frac{2+\gamma}{1+\gamma}} + \mathcal{O}\big(j^2 n^{-\frac{2+\gamma}{1+\gamma}}\big),$$

and $\mu_{bn} = n - \mu_{0b}$. Furthermore,

$$g_{\Theta,b}(r_m) = \sum_{k=b+1}^{\infty} k^{\gamma-1} r_m^k = \sum_{k=1}^{\infty} k^{\gamma-1} r_m^k - \sum_{k=1}^{b} k^{\gamma-1} r_m^k = \mathrm{Li}_{1-\gamma}(r_m) - \sum_{k=1}^{b} k^{\gamma-1} r_m^k,$$

where Li denotes the polylogarithm as in (5.12) and

$$t^k - r_{\widetilde{m}}^k = e^{-k\,\eta_\gamma}\bigg(1 - \exp\Big(\frac{\Gamma(1+\gamma)^{\frac{1}{1+\gamma}}}{1+\gamma} k j n^{-\frac{2+\gamma}{1+\gamma}} + \mathcal{O}(j^2 n^{-\frac{3+2\gamma}{1+\gamma}})\Big)\bigg).$$

Then for $k \le b$ we have $kjn^{-1-\frac{1}{1+\gamma}} = o(1)$ and this yields

$$t^k - r_{\widetilde{m}}^k = e^{-k\,\eta_\gamma}\bigg(- \frac{\Gamma(1+\gamma)^{\frac{1}{1+\gamma}}}{1+\gamma} k j n^{-\frac{2+\gamma}{1+\gamma}} + \mathcal{O}(k j^2 n^{-\frac{3+2\gamma}{1+\gamma}})\bigg).$$

Thus,

$$\sum_{k=1}^{b} k^{\gamma-1}(t^k - r_{\widetilde{m}}^k) = -\frac{\Gamma(1+\gamma)^{\frac{1}{1+\gamma}}}{1+\gamma} jn^{-\frac{2+\gamma}{1+\gamma}} \sum_{k=1}^{b} k^\gamma t^k + \mathcal{O}\bigg(j^2 n^{-\frac{3+2\gamma}{1+\gamma}} \sum_{k=1}^{b} k^\gamma t^k\bigg)$$
$$= -\frac{\Gamma(1+\gamma)^{\frac{1}{1+\gamma}}}{1+\gamma} jn^{-\frac{2+\gamma}{1+\gamma}} \mu_{0b} + \mathcal{O}\bigg(j^2 n^{-\frac{3+2\gamma}{1+\gamma}} \mu_{0b}\bigg),$$

and notice that the error term converges to zero. Altogether, we have proved so far

$$G_{\Theta,b}(r_{\widetilde{m}}, t) = g_{\Theta,b}(r_{\widetilde{m}}) + \widetilde{m}\,(v_{\widetilde{m}} - \eta_\gamma) + \sum_{k=1}^{b} k^{\gamma-1} t^k - \frac{n\,\eta_\gamma}{\gamma} - \zeta(1-\gamma)$$
$$= \mathrm{Li}_{1-\gamma}(r_{\widetilde{m}}) + \widetilde{m}\,(v_{\widetilde{m}} - \eta_\gamma) + \sum_{k=1}^{b} k^{\gamma-1}(t^k - r_{\widetilde{m}}^k) - \frac{n\,\eta_\gamma}{\gamma} - \zeta(1-\gamma)$$
$$= \mathrm{Li}_{1-\gamma}(r_{\widetilde{m}}) - \frac{\Gamma(1+\gamma)^{\frac{1}{1+\gamma}}}{1+\gamma} jn^{-\frac{1}{1+\gamma}} - \frac{n\,\eta_\gamma}{\gamma} - \zeta(1-\gamma) + \mathcal{O}\big(j^2 n^{-\frac{2+\gamma}{1+\gamma}}\big).$$

Finally, we have

$$\mathrm{Li}_{1-\gamma}(r_m) = \Gamma(\gamma) v_m^{-\gamma} + \zeta(1-\gamma) + \mathcal{O}(v_m)$$
$$= \frac{\Gamma(\gamma)}{\Gamma(1+\gamma)^{\frac{\gamma}{1+\gamma}}} \Big(n^{\frac{\gamma}{1+\gamma}} + \frac{\gamma}{1+\gamma} jn^{-\frac{1}{1+\gamma}} + \mathcal{O}(j^2 n^{-\frac{2+\gamma}{1+\gamma}})\Big) + \zeta(1-\gamma)$$

which yields

$$G_{\Theta,b}(r_{\widetilde{m}}, t) = \mathcal{O}\big(j^2 n^{-\frac{2+\gamma}{1+\gamma}}\big) = \mathcal{O}\bigg(\frac{g^2(b)\,\sigma_{0b}^2}{\sigma_{bn}^2}\bigg), \tag{5.28}$$

and this converges to zero because of (5.26). Altogether, we have proved that if $b = o(n^{\frac{1}{1+\gamma}})$ then

$$\mathrm{d}_b(n) = \mathcal{O}\big(b^{2+\gamma} n^{-\frac{2+\gamma}{1+\gamma}} + b^{-\gamma/6} + n^{-\frac{\gamma}{1+\gamma}}\big).$$

To complete the proof of Theorem 5.1, we assume now $b \neq o(n^{\frac{1}{1+\gamma}})$ and show that in this case $\liminf_{n\to\infty} \mathrm{d}_b(n) > 0$. Recall from (5.18) that

$$\mathrm{d}_b(n) \geq \mathbb{P}_\Theta^t[T_{0b} > n].$$

For $b\,n^{-\frac{1}{1+\gamma}} \to \infty$ the mean of T_{0b} is $n + \mathcal{O}(1)$ and the variance is of order $n^{\frac{2+\gamma}{1+\gamma}}$. Thus $\mathbb{P}_\Theta^t[T_{0b} > n] > 0$ for all n. But if $b = cn^{\frac{1}{1+\gamma}}$, then $\mathbb{E}_\Theta^t[T_{0b}] = Cn + \mathcal{O}(1)$ where $C = C(c)$ can be very small when c is very small. In particular, if $C < 1$, then $\mathbb{P}_\Theta^t[T_{0b} > n] \to 0$, thus a more elaborate argument is needed.

A crucial point in the proof above is equation (5.26). Notice that for $b = cn^{\frac{1}{1+\gamma}}$ the usual computations give

$$\mu'_{0b} := \mathbb{E}_\Theta^t[T_{0b}] = \mathcal{O}(n) \quad \text{and} \quad \sigma'_{0b} := \sqrt{\mathbb{V}_\Theta^t[T_{0b}]} = \mathcal{O}\big(n^{\frac{2+\gamma}{2(1+\gamma)}}\big)$$

as well as

$$\mu'_{bn} := \mathbb{E}^t_\Theta\left[T_{bn}\right] = \mathcal{O}(n) \quad \text{and} \quad \sigma'_{bn} := \sqrt{\mathbb{V}^t_\Theta\left[T_{bn}\right]} = \mathcal{O}\big(n^{\frac{2+\gamma}{2(1+\gamma)}}\big).$$

Thus, unlike as in (5.26), here $\sigma'_{0b} = o(\sigma'_{bn})$ does not hold, but we have $\sigma'_{0b} = \mathcal{O}(\sigma'_{bn})$. Therefore, $\mathbb{P}^t_\Theta[T_{bn} = \mu'_{bn} - k]/\mathbb{P}^t_\Theta[T_{0n} = n]$ will not converge to 1 implying that $\mathrm{d}_b(n)$ will not converge to 0. In a different setting, this was also proven in [21]: suppose that $\mathrm{d}_b(n) \to 0$ for $b = cn^{\frac{1}{1+\gamma}}$ with c some non-negative constant. Then the random variables

$$\sum_{m=\frac{c}{2}n^{\frac{1}{1+\gamma}}}^{cn^{\frac{1}{1+\gamma}}} C_m \quad \text{and} \quad \sum_{m=\frac{c}{2}n^{\frac{1}{1+\gamma}}}^{cn^{\frac{1}{1+\gamma}}} Z_m$$

would have same limit as $n \to \infty$. However, it was shown in [21, Theorem 3.6, Theorem 4.6 and Remark 4.4] that for all $c > 0$ these two random variables satisfy two different central limit theorems. □

Remark 5.10. Notice that the term $b^{-\gamma/6}$ in the order of $\mathrm{d}_b(n)$ in Theorem 5.1 comes from the Kolmogorov distance $\mathrm{d}_{\mathrm{K}}(T^x_{0b}, G_b)$. Instead of using the Gaussian approximation, one could prove the first part of the theorem also using saddle-point analysis to compute $\mathbb{P}^t_\Theta\left[T_{0b} = m\right]$ explicitly. This would give the same result but without the $b^{-\gamma/6}$ term. However, we decided to state the proof using the Gaussian approximation since it allows an intuitive understanding why this statement should be true.

5.3 The Erdős-Turán law

In this section we will prove a central limit theorem for $\log O_n$, that is we extend the Erdős-Turán law to permutations with polynomial cycle weights. More precisely, we will prove

Theorem 5.11. *Assume $\theta_m = m^\gamma$ with $0 < \gamma < 1$. Then, as $n \to \infty$,*

$$\frac{\log O_n - G(n)}{\sqrt{F(n)}} \xrightarrow{d} \mathcal{N}(0,1)$$

holds, where

$$F(n) = \frac{K(\gamma)}{(1+\gamma)^3} n^{\frac{\gamma}{1+\gamma}} \log^2(n),$$
$$G(n) = \frac{K(\gamma)}{1+\gamma} n^{\frac{\gamma}{1+\gamma}} \log(n) + n^{\frac{\gamma}{1+\gamma}} H(\gamma) \quad \text{with}$$
$$H(\gamma) = K(\gamma)\left(\frac{\Gamma'(\gamma)}{\Gamma(\gamma)} - \frac{\log(\Gamma(1+\gamma))}{1+\gamma}\right);$$

here Γ' *denotes the derivative of the gamma function and*

$$K(\gamma) = \Gamma(\gamma)\Gamma(1+\gamma)^{-\frac{\gamma}{1+\gamma}}.$$

The strategy to prove this theorem is as follows. First, we show that $\hat{g}_\Theta(t,s)$ as in (5.11) is log-admissible. Then Lemma 5.4 yields a precise expansion of the generating function of $\log Y_n$ from which we deduce the required central limit theorem for $\log Y_n$. The complicated part is to transfer the result to $\log O_n$; in Lemma 5.17 we prove that $\log Y_n$ and $\log O_n$ are sufficiently close for $0 < \gamma < 1$.

Lemma 5.12. *$\hat{g}_\Theta(t,s)$ is* log*-admissible for $\gamma > 0$, $s > -\alpha$.*

Proof. For $k \geq 1$ as $t \to 1$ holds

$$\hat{g}_\Theta^{(k)}(t) = t^{-k}\operatorname{Li}_{a-k}(t) = \Gamma(1+k-a)(-\log(t))^{a-k-1}t^{-k} + \mathcal{O}(1). \tag{5.29}$$

The proof that $\hat{g}_\Theta(t,s)$ satisfies the properties given in Definition 5.3 is analogous to the proof of Proposition 3.7 in [57]; one simply has to verify that all involved expressions are uniform in s for $-\gamma + \epsilon \leq s \leq C$ for some constant C. This is straightforward and we thus omit the details. $\square$

Let us now compute the generating function of $\log Y_n$ by means of Lemma 5.4.

Proposition 5.13. *Let $\hat{g}_\Theta$ be as in (5.11) with $\gamma > 0$. Then we have*

$$\begin{aligned}\mathbb{E}_\Theta[\exp(s\log Y_n)] = &\Big(\sqrt{\tilde{\gamma}_{2,s}}\, n^{\frac{1}{2}(\frac{1}{1+\gamma}-\frac{1}{1+\gamma+s})}\Big)\exp\left(\tilde{\gamma}_{1,s}\, n^{1-\frac{1}{1+\gamma+s}} - \tilde{\gamma}_{1,0}\, n^{1-\frac{1}{1+\gamma}}\right)\\ &\times \exp\big(\zeta(1-s-\gamma)-\zeta(1-\gamma)\big)\big(1+o(1)\big)\end{aligned}$$

with

$$\tilde{\gamma}_{1,s} = \frac{(1+\gamma+s)\Gamma(\gamma+s)}{\Gamma(1+\gamma+s)^{1-\frac{1}{1+\gamma+s}}}, \quad \tilde{\gamma}_{2,s} = \frac{(1+\gamma)\Gamma(1+\gamma+s)^{\frac{1}{1+\gamma+s}}}{(1+\gamma+s)\Gamma(1+\gamma)^{\frac{1}{1+\gamma}}},$$

where the error bounds are uniform in s for bounded s, $s > -\gamma + \epsilon$.

Proof. We first compute r_{ns}. This should satisfy

$$\alpha_s(r_{ns}) = n$$

but as stated in Remark 5.5 actually it suffices that

$$\alpha_s(r_{ns}) - n = o\Big(\sqrt{\beta_s(r_{ns})}\Big) \tag{5.30}$$

holds. We set for $a = 1 - s - \gamma$

$$r_{ns} = \exp\left(-\Big(\frac{n}{\Gamma(2-a)}\Big)^{\frac{1}{a-2}}\right)$$

and obtain

$$\alpha_s(r_{ns}) = n + \mathcal{O}(1) \quad \text{and} \quad \beta_s(r_{ns}) = \Gamma(3-a)\Big(\frac{n}{\Gamma(2-a)}\Big)^{1+\frac{1}{2-a}} + \mathcal{O}(1),$$

so that (5.30) holds. Furthermore,

$$\hat{g}_\Theta(r_{ns}, s) = \Gamma(1-a)\Big(\frac{n}{\Gamma(2-a)}\Big)^{1-\frac{1}{2-a}} + \zeta(1-s-\gamma) + o(1).$$

We now have

$$G_{n,s} = [t]^n \exp(\hat{g}_\Theta(t,s)) = h_n \mathbb{E}_\Theta[\exp(s \log Y_n)]$$

Therefore,

$$\begin{aligned} h_n = G_{n,0} &= \frac{1}{\sqrt{2\pi}}\big(r_{n0}\big)^{-n}\beta_0(r_{n0})^{-1/2}\exp\big(\hat{g}_\Theta(r_{n0},0)\big)\big(1+o(1)\big) \\ &= \big(2\pi\Gamma(2+\gamma)\big)^{-\frac{1}{2}}\Big(\frac{\Gamma(1+\gamma)}{n}\Big)^{\frac{2+\gamma}{2(1+\gamma)}} \times \\ &\quad \exp\left(\frac{1+\gamma}{\gamma}\,\Gamma(1+\gamma)^{\frac{1}{1+\gamma}}\, n^{\frac{\gamma}{1+\gamma}} + \zeta(1-\gamma)\right)\big(1+o(1)\big). \end{aligned}$$

and

$$\mathbb{E}_\Theta[\exp(s\log Y_n)] = \left(\frac{r_{n0}}{r_{ns}}\right)^n \left(\frac{\beta_0(r_{n0})}{\beta_s(r_{ns})}\right)^{1/2} \exp\big(\hat{g}_\Theta(r_{ns},s) - \hat{g}_\Theta(r_{n0},0)\big)\big(1+o(1)\big)$$

and this gives the result. □

Remark 5.14. Given Theorem 5.1, a natural way to investigate further properties of $\log O_n$, for example to prove the central limit theorem, would be to work with the functional $\log P_n := \sum_{m=1}^n Z_m \log(m)$ instead of with $\log Y_n = \sum_{m=1}^n C_m \log(m)$ and to show that the contribution of the large components $C_{b+1}, \dots, C_n$ is negligible. However, in the current setting, the large cycle counts actually do contribute to the behavior of $\log O_n$. To see this, one may easily compute the moment generating function of $\log P_n$ to show that it satisfies the central limit theorem

$$\frac{\log P_n - \tilde{G}(n)}{\sqrt{F(n)}} \xrightarrow{d} \mathcal{N}(0,1)$$

where $F(n)$ is as in Theorem 5.11 but

$$\tilde{G}(n) = \frac{K(\gamma)}{1+\gamma}\, n^{\frac{\gamma}{1+\gamma}} \log(n) + n^{\frac{\gamma}{1+\gamma}}\, \tilde{H}(\gamma) \tag{5.31}$$

with

$$\tilde{H}(\gamma) = -K(\gamma)\frac{\log\Gamma(1+\gamma)}{1+\gamma}.$$

Thus, even rescaled by $F(n)$, the discrepancy between $G(n)$ and $\tilde{G}(n)$ is too large to prove the central limit theorem for $\log O_n$ via the independent approximating process. More generally, it seems that the bound $b = o(n^{\frac{1}{1+\gamma}})$ is too small to exploit Theorem 5.1 to study the whole cycle count process. Nonetheless, in Section 5.4 we will explain how to use Theorem 5.1 in order to investigate properties of the small cycles.

With Proposition 5.13 at hand, we can establish the Erdős-Turán law for $\log Y_n$.

Corollary 5.15. *Let $\hat{g}_\Theta$ be as in (5.11) with $\gamma > 0$. Then we have, as $n \to \infty$,*

$$\frac{\log Y_n - G(n)}{\sqrt{F(n)}} \xrightarrow{d} \mathcal{N}(0,1)$$

where $F(n)$ and $G(n)$ are as in Theorem 5.11.

Remark 5.16. Landau's result (4.1) implies immediately that an analogous result to Corollary 5.15 for $\log O_n$ can be valid only for $0 < \gamma < 1$. This means that $\log Y_n$ is a good approximation for $\log O_n$ when $0 < \gamma < 1$ but for $\gamma \geq 1$ the behavior of the two random variables is indeed different.

Proof of Corollary 5.15. We write the the expansion in Proposition 5.13 as

$$\mathbb{E}_\Theta[\exp(s \log Y_n)] = \exp(f(n,s))$$

and expand the function $f(n,s)$ around $s = 0$. Now set $F(n)$ as in Theorem 5.11. Since the error terms in Proposition 5.13 are uniform in s we can apply it for $s/\sqrt{F(n)}$.Then, as $n \to \infty$,

$$\mathbb{E}_\Theta\left[\exp\left(s\frac{\log Y_n}{\sqrt{F(n)}}\right)\right] \sim \exp\left(\frac{s^2}{2} + \bar{G}(n)H(\gamma)s\right)$$

where $H(\gamma)$ is defined in Theorem 5.11 and

$$\bar{G}(n) = \sqrt{\Gamma(\gamma) + \Gamma(1+\gamma)}\left(\frac{n}{\Gamma(1+\gamma)}\right)^{\frac{\gamma}{2(1+\gamma)}} \log\left(\frac{n}{\Gamma(1+\gamma)}\right).$$

By means of Lévy's continuity theorem the result follows. □

To transfer the result from $\log Y_n$ to $\log O_n$ we need to show that they are close in a certain sense. We will prove the following

Lemma 5.17. *For $\theta_m = m^\gamma$ with $0 < \gamma < 1$ the following holds as $n \to \infty$:*

$$\mathbb{P}_\Theta\big(\log Y_n - \log O_n \geq \log(n) \log\log(n)\big) \to 0.$$

Proof. First, recall (5.9) and (5.10) and notice that

$$\log O_n = \psi(n) - R(n)$$

where

$$\psi(n) = \sum_{k=1}^{n} \Lambda(k) \quad \text{and} \quad R(n) = \sum_{k=1}^{n} \Lambda(k)\, \mathbb{1}_{\{D_{nk}=0\}}\,.$$

Recall that ψ is the so-called Chebyshev function which satisfies the asymptotic (2.23), that is $\psi(n) = n(1+o(1))$. We need to identify the smallest b such that for $g(n) = \log(n)\log\log(n)$

$$\mathbb{P}_\Theta\Big(\log Y_n - \psi(b) \geq \frac{g(n)}{2}\Big) \to 0 \qquad \text{as } n \to \infty. \tag{5.32}$$

Corollary 5.15 implies that

$$\mathbb{P}_\Theta\Big(\log Y_n - h(n) \geq \epsilon\Big) \to 0 \qquad \text{as } n \to \infty,$$

for any $\epsilon > 0$ and functions h such that $h(n)/n^{\frac{\gamma}{1+\gamma}}\log(n) \to \infty$. Therefore, choose $b = n^{\frac{\gamma}{1+\gamma}}\log^2(n)$, then (5.32) is satisfied (actually, it holds for any positive function $g(n)$). It remains to prove

$$\mathbb{P}_\Theta\Big(R(n) - \sum_{k=b+1}^{n} \Lambda(k) \geq \frac{g(n)}{2}\Big) \to 0 \qquad \text{as } n \to \infty. \tag{5.33}$$

Notice that

$$R(n) - \sum_{k=b+1}^{n} \Lambda(k) \leq R(b) \leq \sum_{k=1}^{b} \Lambda(k)\, \mathbb{1}_{\{C_k=0\}} =: S(b)$$

and thus it suffices to show

$$\mathbb{P}_\Theta\Big(S(b) \geq \frac{g(n)}{2}\Big) \to 0 \qquad \text{as } n \to \infty.$$

To prove this we will approximate $S(b)$ by the functional

$$S'(b) := \sum_{k=1}^{b} \Lambda(k)\, \mathbb{1}_{\{Z_k=0\}}$$

where the Z_k are independent Poisson random variables with parameter $k^{\gamma-1}t^k$ as in (5.15). Then

$$\mathbb{P}_\Theta\Big(S(b) \geq \frac{g(n)}{2}\Big) = \mathbb{P}_\Theta\Big(S'(b) \geq \frac{g(n)}{2}\Big) + \mathcal{O}\big(\,\mathrm{d_K}(S(b), S'(b))\big),$$

where $\mathrm{d_K}(X, Y)$ denotes again the Kolmogorov distance of the random variables X and Y; see (2.20). Clearly,

$$\mathrm{d_K}(S(b), S'(b)) \le \mathrm{d_{TV}}(S(b), S'(b)) \le \mathrm{d}_b(n),$$

and Theorem 5.1 shows that $\mathrm{d}_b(n) \to 0$ if and only if $b = o(n^{\frac{1}{1+\gamma}})$. For $0 < \gamma < 1$ we have $b = n^{\frac{\gamma}{1+\gamma}} \log^2(n) = o(n^{\frac{1}{1+\gamma}})$. Therefore, it suffices to show

$$\mathbb{P}_\Theta\Big(S'(b) \ge \frac{g(n)}{2}\Big) \to 0 \qquad \text{as } n \to \infty,$$

which is for $s \ge 0$ equivalent to

$$\log \mathbb{P}_\Theta\Big(e^{sS'(b)} \ge e^{\frac{sg(n)}{2}}\Big) \to -\infty \qquad \text{as } n \to \infty. \tag{5.34}$$

The moment generating function of $S'(b)$ is given by

$$\mathbb{E}_\Theta\left[e^{sS'(b)}\right] = \prod_{k=1}^{b} \Big(1 + e^{-k^{\gamma-1}t^k}\big(e^{s\Lambda(k)} - 1\big)\Big),$$

where $t^k = \exp(-kn^{-\frac{1}{1+\gamma}})$. By Markov's inequality and with $\log(1+z) \le z$ we get

$$\begin{aligned}
\log \mathbb{P}_\Theta\Big(e^{sS'(b)} \ge e^{\frac{sg(n)}{2}}\Big) &\le -\frac{sg(n)}{2} + \sum_{k=1}^{b} \log\Big(1 + e^{-t^k k^{\gamma-1}}\big(e^{s\Lambda(k)} - 1\big)\Big) \\
&\le -\frac{sg(n)}{2} + \sum_{k=1}^{b} e^{-t^k k^{\gamma-1}}\big(e^{s\Lambda(k)} - 1\big) \\
&\le -\frac{sg(n)}{2} + \big(e^{s\log(n)} - 1\big) \sum_{k=1}^{b} e^{-t^k k^{\gamma-1}}.
\end{aligned}$$

Since $b = o\big(n^{\frac{1}{1+\gamma}}\big)$, there is a constant $c > 0$ such that

$$\begin{aligned}
\sum_{k=1}^{b} \exp\big(-t^k k^{\gamma-1}\big) &\le \sum_{k=1}^{b} \exp\big(-k^{\gamma-1}\exp(-bn^{-\frac{1}{1+\gamma}})\big) = \sum_{k=1}^{b} \exp\big(-ck^{\gamma-1}\big) \\
&\le \int_1^b \exp\big(-cx^{\gamma-1}\big)dx = \mathcal{O}\Big(\Gamma\Big(\frac{1}{1-\gamma}, b^{\gamma-1}\Big) - \Gamma\Big(\frac{1}{1-\gamma}, 1\Big)\Big) \\
&= \mathcal{O}(1).
\end{aligned}$$

Thus for $g(n) = \log(n)\log\log(n)$ and $s := \sqrt{\log\log(n)}/g(n)$ this yields

$$\log \mathbb{P}_\Theta\Big(e^{sS'(b)} \ge e^{\frac{s}{2}}\Big) \le -\frac{\sqrt{\log\log(n)}}{2} + \mathcal{O}\big(\log\log(n)^{-1/2}\big).$$

The proof is complete. □

Proof of Theorem 5.11. The statement of Theorem 5.11 is a direct consequence of Corollary 5.15 and Lemma 5.17. □

5.4 A functional central limit theorem

In Remark 5.14 was mentioned that the bound $b = o(n^{\frac{1}{1+\gamma}})$ is too small to study properties of the whole cycle count process via the independent approximating Poisson random variables. However, here we give an example how to exploit Theorem 5.1 in order to study the behavior of the small cycles. For $x > 0$ we define $x^* := \lfloor x\, n^{\frac{\gamma}{1+\gamma}} \rfloor$ and

$$B_n(x) := \frac{\log O_{x^*} - \frac{1}{1+\gamma}\, x^\gamma \log(n)\, n^{\frac{\gamma^2}{1+\gamma}}}{\frac{\sqrt{\gamma}}{1+\gamma} \log(n)\, n^{\frac{\gamma^2}{2(1+\gamma)}}}, \tag{5.35}$$

where $O_{x^*} := \operatorname{lcm}\{m \leq x^* \,;\, C_m > 0\}$. We will prove the following

Theorem 5.18. *Let $\hat{g}_\Theta$ be as in (5.11) with $0 < \gamma < 1$, $B_n(x)$ as in* (5.35) *and denote by $\mathcal{W}$ a standard Brownian motion. Then, as $n \to \infty$ and for $x > 0$, $B_n(t)$ converges weakly to $\mathcal{W}(x^\gamma)$.*

Proof. First notice that

$$\log Y_{x^*} - \log O_{x^*} \leq \log Y_n - \log O_n$$

and thus by means of Lemma 5.17 it is sufficient to show that

$$\frac{\log Y_{x^*} - \frac{1}{1+\gamma}\, x^\gamma \log(n)\, n^{\frac{\gamma^2}{1+\gamma}}}{\frac{\sqrt{\gamma}}{1+\gamma} \log(n)\, n^{\frac{\gamma^2}{2(1+\gamma)}}}$$

satisfies the required convergence. Since $x^* = o(n^{\frac{1}{1+\gamma}})$ and in a discrete probability space $d_b(n) \to 0$ is equivalent to convergence in distribution of $(C_1, \ldots, C_b)$ to $(Z_1, \ldots, Z_b)$, Theorem 5.1 yields

$$\mathbb{E}_\Theta\left[e^{s \log Y_{x^*}}\right] = \mathbb{E}_\Theta\left[e^{s \log P_{x^*}}\right]\big(1 + o(1)\big)$$

where $\log P_{x^*} = \sum_{m=1}^{x^*} Z_m \log(m)$. Thus we have to show that, as $n \to \infty$,

$$\widetilde{B}_n(x) := \frac{\log P_{x^*} - \frac{1}{1+\gamma}\, x^\gamma \log(n)\, n^{\frac{\gamma^2}{1+\gamma}}}{\frac{\sqrt{\gamma}}{1+\gamma} \log(n)\, n^{\frac{\gamma^2}{2(1+\gamma)}}} \xrightarrow{d} \mathcal{W}(x^\gamma).$$

The convergence of the finite dimensional distributions is easily established. By independence, the characteristic function of $\log P_{x^*}$ is given by

$$\mathbb{E}_\Theta\left[e^{is \log P_{x^*}}\right] = \exp\left(\sum_{m=1}^{x^*} m^{\gamma-1} t^m (e^{is\log(m)} - 1)\right) = \exp\left(is\alpha(x^*) - \frac{s^2}{2}\beta(x^*) + \delta(s, x^*)\right)$$

where

$$\alpha(x^*) = \sum_{m=1}^{x^*} m^{\gamma-1} t^m \log(m), \quad \beta(x^*) = \sum_{m=1}^{x^*} m^{\gamma-1} t^m \log^2(m),$$

and

$$\delta(s,x^*) = \sum_{j=3}^{\infty} \sum_{m=1}^{x^*} m^{\gamma-1} t^m \frac{(is)^j \log^j(m)}{j!}.$$

With computations similar to those in the proof of Lemma 5.8 we get

$$\begin{aligned}
\alpha(x^*) &= \frac{1}{1+\gamma}\, x^\gamma\, n^{\frac{\gamma^2}{1+\gamma}} \log(n) + \mathcal{O}\big(n^{\frac{\gamma^2}{1+\gamma}}\big),\\
\beta(x^*) &= \frac{\gamma}{(1+\gamma)^2} x^\gamma\, n^{\frac{\gamma^2}{1+\gamma}} \log^2(n) + \mathcal{O}\big(n^{\frac{\gamma^2}{1+\gamma}} \log(n)\big),\\
\sum_{m=1}^{x^*} m^{\gamma-1} t^m \log^3(m) &= \mathcal{O}\big(n^{\frac{\gamma^2}{1+\gamma}} \log(n)\big).
\end{aligned}$$

This proves that for every fixed x we have, as $n \to \infty$,

$$\widetilde{B}_n(x) \xrightarrow{d} \mathcal{N}(0, x^\gamma).$$

It remains to prove that the process $\widetilde{B}_n$ is tight. We use the moment condition given in [20, Theorem 15.6], that is we have to show that for any $n \geq 0$ and $0 \leq x_1 < x < x_2$

$$\mathcal{E}_\Theta(x_1,x_2) := \mathbb{E}_\Theta\left[(\widetilde{B}_n(x) - \widetilde{B}_n(x_1))^2 (\widetilde{B}_n(x_2) - \widetilde{B}_n(x))^2\right] = \mathcal{O}\big((x_2-x_1)^2\big).$$

To prove this we use the independence of the Z_m. Denote

$$\log P_{x^*}^{y^*} = \sum_{m=x^*+1}^{y^*} Z_m \log(m).$$

Then

$$\begin{aligned}
\mathcal{E}_\Theta(x_1,x_2) &= \mathcal{O}\bigg(\Big(x^\gamma\, n^{\frac{\gamma^2}{1+\gamma}} \log^2(n)\Big)^{-2} \mathbb{V}_\Theta\big(\log P_{x_1^*}^{x^*}\big) \mathbb{V}_\Theta\big(\log P_{x^*}^{x_2^*}\big)\bigg)\\
&= \mathcal{O}\bigg(\Big(x^\gamma\, n^{\frac{\gamma^2}{1+\gamma}} \log^2(n)\Big)^{-2} \big(\beta(x^*) - \beta(x_1^*)\big)\big(\beta(x_2^*) - \beta(x^*)\big)\bigg)\\
&= \mathcal{O}\big((x^\gamma - x_1^\gamma)(x_2^\gamma - x^\gamma)\big) = \mathcal{O}\big((x_2-x_1)^2\big).
\end{aligned}$$

This completes the proof. □

5.5 Large deviations estimates

In this section we will prove the precise large deviations estimate stated in Theorem 5.2. To this end, we will show that $\log Y_n$, appropriately rescaled, is mod-Poisson convergent. However, first we show how one can deduce from the moment generating function of $\log Y_n$ given in Proposition 5.13 a classical large deviations result for $\log O_n$. We will show that for any Borel set B

$$\lim_{n\to\infty} n^{-\frac{\gamma}{1+\gamma}} \log \mathbb{P}_\Theta\left(\frac{\log O_n}{n^{\frac{\gamma}{1+\gamma}}\log(n)} \in B\right) = -\inf_{x\in B} \chi^*(x)$$

holds, where

$$\chi^*(x) = \sup_{t\in\mathbb{R}}[tx - \chi(t)]$$

is the so-called Fenchel-Legendre transform of $\chi(t)$ given by

$$\chi(t) := \frac{(1+\gamma)\Gamma(\gamma)}{\Gamma(1+\gamma)^{\frac{\gamma}{1+\gamma}}}\big(e^{\frac{t}{(1+\gamma)^2}} - 1\big). \tag{5.36}$$

In other words, we will prove the following

Theorem 5.19. *Let $\hat{g}_\Theta$ be defined as in (5.11) with $0 < \gamma < 1$. The sequence $\log O_n/n^{\frac{\gamma}{1+\gamma}}\log(n)$ satisfies a large deviations principle with rate $n^{\frac{\gamma}{1+\gamma}}$ and rate function given by the convex dual of $\chi(t)$ defined in (5.36).*

Proof. Let us first check that $\log Y_n/n^{\frac{\gamma}{1+\gamma}}\log(n)$ satisfies this large deviations estimate. By the Gärtner-Ellis theorem it suffices to prove

$$\lim_{n\to\infty} n^{-\frac{\gamma}{1+\gamma}} \log \mathbb{E}_\Theta\left[\exp\left(t\,\frac{\log Y_n}{\log(n)}\right)\right] = \chi(t). \tag{5.37}$$

In view of Proposition 5.13 we have to show that for $t^* = t/\log(n)$

$$\lim_{n\to\infty} n^{-\frac{\gamma}{1+\gamma}}\left(\tilde{\gamma}_{1,t^*}\, n^{1-\frac{1}{1+\gamma+t^*}} - \tilde{\gamma}_{1,0}\, n^{1-\frac{1}{1+\gamma}}\right) = \chi(t)$$

holds with

$$\tilde{\gamma}_{1,t} = \frac{(1+\gamma+t)\Gamma(\gamma+t)}{\Gamma(1+\gamma+t)^{1-\frac{1}{1+\gamma+t}}}.$$

This is true since $\Gamma(a+x) = \Gamma(\gamma) + \mathcal{O}(x)$ as $x \to 0$ and therefore

$$\tilde{\gamma}_{1,t^*} = \tilde{\gamma}_{1,0} + \mathcal{O}\big(\log^{-1}(n)\big), \tag{5.38}$$

$$\begin{aligned} n^{1-\frac{1}{1+\gamma+t^*}} &= n^{1-\frac{1}{1+\gamma}}\left(1 + \sum_{k=1}^{\infty}\frac{t^k}{(1+\gamma)^{2k}k!} + \mathcal{O}\big(\log^{-1}(n)\big)\right) \\ &= n^{\frac{\gamma}{1+\gamma}}\left(e^{\frac{t}{(1+\gamma)^2}} + \mathcal{O}\big(\log^{-1}(n)\big)\right). \end{aligned} \tag{5.39}$$

Similar the proof of Theorem 4.15, it remains to show that $\log Y_n/n^{\frac{\gamma}{1+\gamma}}\log(n)$ and $\log O_n/n^{\frac{\gamma}{1+\gamma}}\log(n)$ are exponentially equivalent with rate $n^{\frac{\gamma}{1+\gamma}}$. This is subject of the following lemma. $\square$

Lemma 5.20. *Let $\hat{g}_\Theta$ be as in (5.11) with $0<\gamma<1$, then for any $c>0$ the following holds:*

$$\limsup_{n\to\infty} n^{-\frac{\gamma}{1+\gamma}}\log\mathbb{P}_\Theta\left[\log Y_n-\log O_n>c\,n^{\frac{\gamma}{1+\gamma}}\log(n)\right]=-\infty.$$

Proof. We will prove a stronger version of this asymptotic in Lemma 5.21 below. $\square$

The statement of Theorem 5.19 can be refined. Recall the notion of mod-φ convergence which was explained in Section 2.3. Here, we notice first that mod-Poisson convergence holds for $\log Y_n$, appropriately rescaled. Indeed, from the moment generating function of $\log Y_n$ in Proposition 5.13 combined with (5.38) and (5.39) we deduce that

$$\mathcal{Y}_n:=\frac{(1+\gamma)^2\log Y_n}{\log(n)}$$

is mod-Poisson convergent with parameter

$$\lambda_n:=\frac{(1+\gamma)\Gamma(\gamma)}{\Gamma(1+\gamma)^{\frac{\gamma}{1+\gamma}}}\,n^{\frac{\gamma}{1+\gamma}}\big(1+\mathcal{O}(\log^{-1}(n))\big).$$

Notice that the constant factor comes from $\tilde{\gamma}_{1,0}$ in Proposition 5.13. This convergence is surprising since the rescaling of $\log(n)$ in $\mathcal{Y}_n$ is relatively insignificant compared to the order of $\log Y_n$ which is $n^{\frac{\gamma}{1+\gamma}}\log(n)$. Therefore, this statement suggest that $\log Y_n$ is indeed close to a Poisson random variable. However, the rescaling is too small to deduce that also the distribution of $\log O_n/\log(n)$ is close to a Poisson distribution.

Now, let us apply Theorem 2.22 in order to prove the precise large deviations estimate for $\log O_n$ which was stated in Theorem 5.2. First, since Theorem 2.22 requires mod-φ convergence where the reference law is lattice distributed, we cannot work directly with the mod-Poisson convergence of $\mathcal{Y}_n$. However, recall (2.21), thus

$$\widetilde{\mathcal{Y}}_n:=\frac{\mathcal{Y}_n-\lambda_n}{\lambda_n^{1/3}}$$

is mod-$\mathcal{N}(0,\lambda_n^{1/3})$ convergent with limiting function $\Phi(t)=e^{t^3/6}$. Then define

$$\Omega_n:=\frac{\log O_n-\lambda_n\log(n)(1+\gamma)^{-2}}{\lambda_n^{1/3}\log(n)(1+\gamma)^{-2}}.$$

We have to prove that for $0<\gamma<1$ and for any $x>0$ the asymptotic

$$\mathbb{P}_\Theta\left[\Omega_n\ge x\lambda_n^{1/3}\right]=\frac{\exp(-\lambda_n^{1/3}\frac{x^2}{2}+\frac{x^3}{6})}{x\lambda_n^{1/6}\sqrt{2\pi}}\,(1+o(1))$$

holds.

Proof of Theorem 5.2. Let us first check that $\widetilde{\mathfrak{Y}}_n$ satisfies the required precise deviations estimate. Similarly to the proof of Theorem 4.17, this follows from Theorem 2.22 with $\beta_n = \lambda_n^{1/3}$, $F(x) = x^2/2 = \eta(x)$ and $h(x) = x$. It remains to transfer the estimate to Ω_n. Clearly,

$$\mathbb{P}_\Theta\left[\Omega_n \geq x\lambda_n^{1/3}\right] \leq \mathbb{P}_\Theta\left[\widetilde{\mathfrak{Y}}_n \geq x\lambda_n^{1/3}\right].$$

For the other direction, let g be a positive function such that $g(n) = o(\lambda_n^{1/3})$. Then

$$\mathbb{P}_\Theta\left[\widetilde{\mathfrak{Y}}_n \geq x\lambda_n^{1/3} + g(n)\right] \leq \mathbb{P}_\Theta\left[\Omega_n \geq x\lambda_n^{1/3}\right] + \mathbb{P}_\Theta\left[\Delta_n \geq g(n)\lambda_n^{1/3}\log(n)\right]$$

holds and we also have

$$\mathbb{P}_\Theta\left[\widetilde{\mathfrak{Y}}_n \geq x\lambda_n + g(n)\right] = \mathbb{P}_\Theta\left[\widetilde{\mathfrak{Y}}_n \geq x\lambda_n\right]\big(1 + o(1)\big).$$

Finally, to complete the proof we need to find an appropriate $g(n) = o(\lambda_n^{1/3})$ such that

$$\lim_{n\to\infty} \lambda_n^{-1/3} \log \mathbb{P}_\Theta\left[\Delta_n \geq g(n)\lambda_n^{1/3}\log(n)\right] = -\infty.$$

The following lemma proves that this holds for $g(n) = n^{\frac{\gamma}{3(1+\gamma)}}/\sqrt{\log(n)}$. □

Lemma 5.21. *Let $\hat{g}_\Theta$ be as in (5.11) with $0 < \gamma < 1$, then for any $c > 0$ the following holds:*

$$\lim_{n\to\infty} n^{-\frac{\gamma}{3(1+\gamma)}} \log \mathbb{P}_\Theta\left[\log Y_n - \log O_n > c\, n^{\frac{2\gamma}{3(1+\gamma)}}\sqrt{\log(n)}\right] = -\infty.$$

Proof. The proof is very similar to the proof of Lemma 5.17. Recall (5.9) and (5.10) and notice that

$$\log O_n = \psi(n) - R(n)$$

where

$$\psi(n) = \sum_{k=1}^n \Lambda(k) \quad \text{and} \quad R(n) = \sum_{k=1}^n \Lambda(k)\, \mathbb{1}_{\{D_{nk}=0\}}.$$

Recall that ψ is the so-called Chebyshev function as defined in (2.22) which satisfies the asymptotic (2.23). First, we want to find the smallest b such that

$$n^{-\frac{\gamma}{3(1+\gamma)}} \log \mathbb{P}_\Theta\left[\log Y_n - \psi(b) > \frac{c}{2} n^{\frac{2\gamma}{3(1+\gamma)}}\sqrt{\log(n)}\right] \to -\infty \tag{5.40}$$

and afterwards we show

$$n^{-\frac{\gamma}{3(1+\gamma)}} \log \mathbb{P}_\Theta\left[R(n) - \sum_{k=b+1}^n \Lambda(k) > \frac{c}{2} n^{\frac{2\gamma}{3(1+\gamma)}}\sqrt{\log(n)}\right] \to -\infty. \tag{5.41}$$

Corollary 5.15 tells us that that for

$$x = \frac{\frac{c}{2} n^{\frac{2\gamma}{3(1+\gamma)}} \sqrt{\log(n)} + \psi(b) - G(n)}{\sqrt{F(n)}}$$

we get as $n \to \infty$

$$\mathbb{P}_\Theta \left[\log Y_n - \psi(b) \geq \frac{c}{2} n^{\frac{2\gamma}{3(1+\gamma)}} \sqrt{\log(n)} \right] = \left(1 - \frac{1}{2}\left(1 + \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)\right)\right)(1 + o(1)).$$

Here, erf denotes the error function which satisfies the asymptotic

$$\operatorname{erf}(x) = 1 + \mathcal{O}(x^{-1} e^{-x^2}) \qquad \text{as } x \to \infty.$$

Thus set $b = n^{\frac{\gamma}{1+\gamma}} \log(n) \alpha(n)$ for some function $\alpha \to \infty$ so that

$$x = \mathcal{O}\big(n^{\frac{\gamma}{2(1+\gamma)}} \alpha(n)\big)$$

where the error term has a positive sign. This implies

$$n^{-\frac{\gamma}{3(1+\gamma)}} \log \mathbb{P}_\Theta \left[\log Y_n - \psi(b) > \frac{c}{2} n^{\frac{2\gamma}{3(1+\gamma)}} \sqrt{\log(n)} \right] = \mathcal{O}\Big(n^{-\frac{\gamma}{1+\gamma}} \log\big(x^{-1} e^{-x^2}\big)\Big)$$

which converges indeed to $-\infty$ and hence (5.40) holds. So let us now prove (5.41). Notice that

$$R(n) - \sum_{k=b+1}^{n} \Lambda(k) \leq R(b) \leq \sum_{k=1}^{b} \Lambda(k)\, \mathbb{1}_{\{C_k = 0\}} =: S(b)$$

and therefore

$$n^{-\frac{\gamma}{3(1+\gamma)}} \log \mathbb{P}_\Theta \left[S(b) > \frac{c}{2} n^{\frac{2\gamma}{3(1+\gamma)}} \sqrt{\log(n)} \right] \to -\infty$$

implies (5.41). Via saddle-point analysis we get

$$\mathbb{E}_\Theta \left[e^{sS(b)} \right] = \exp\left(\sum_{k=1}^{b} \log\big(1 + (e^{s\Lambda(k)} - 1) e^{-k^{\gamma-1} t^k}\big) \right) \big(1 + o(1)\big)$$

with t^k as in (5.19). We proceed as in the proof of Lemma 5.17. For any $s \geq 0$ Markov's inequality yields

$$\begin{aligned}
&n^{-\frac{\gamma}{3(1+\gamma)}} \log \mathbb{P}_\Theta \left[S(b) > \frac{c}{2} n^{\frac{2\gamma}{3(1+\gamma)}} \sqrt{\log(n)} \right] \\
&\leq -\frac{sc}{2} n^{\frac{\gamma}{3(1+\gamma)}} \sqrt{\log(n)} + n^{-\frac{\gamma}{3(1+\gamma)}} \log \mathbb{E}_\Theta \left[e^{sS(b)} \right] \\
&\leq -\frac{sc}{2} n^{\frac{\gamma}{3(1+\gamma)}} \sqrt{\log(n)} + n^{-\frac{\gamma}{3(1+\gamma)}} \sum_{k=1}^{b} (e^{s\Lambda(k)} - 1) e^{-k^{\gamma-1} t^k} \\
&= -\frac{sc}{2} n^{\frac{\gamma}{3(1+\gamma)}} \sqrt{\log(n)} + \mathcal{O}\big(n^{-\frac{\gamma}{3(1+\gamma)}} (e^{s\log(n)} - 1)\big).
\end{aligned}$$

For the last equality we need $b = o\left(n^{\frac{1}{1+\gamma}}\right)$ and hence $0 < \gamma < 1$. Now set $s = \log^{-1/2}(n)$ to get

$$n^{-\frac{\gamma}{3(1+\gamma)}} \log \mathbb{P}_\Theta \left[S(b) > \frac{c}{2} n^{\frac{2\gamma}{3(1+\gamma)}} \log(n) \right] \leq -\frac{c}{2} n^{\frac{\gamma}{3(1+\gamma)}} + \mathcal{O}\left(n^{-\frac{\gamma}{1+\gamma}} e^{\log^{1/2}(n)} \right).$$

The proof is complete. $\square$

Appendix

Let us prove here the moment condition which was used in Section 4.4 to establish the large deviation estimates for $m = 2$.

Proposition 1. *Let g_Θ belong to $\mathcal{F}(r, \vartheta, K)$ and $\theta_m r^m = \vartheta + \mathcal{O}(m^{-\delta})$ for some $\delta > 0$. Define*

$$\Delta_{n,\beta(n)} := \sum_{k=\beta(n)}^{n} \Lambda(k)(\widetilde{D}_{nk} - \widetilde{D}^*_{nk}),$$

where $\beta(n) = o(n)$ is such that $\beta(n)/\log^3(n) \geq 1$. We then have for n large enough

$$\mathbb{E}_\Theta\left[(\Delta_{n,\beta(n)})^2\right] \leq (\log(n)\log\log(n))^2.$$

Clearly,

$$\mathbb{E}_\Theta\left[(\Delta_{n,\beta(n)})^2\right] = \sum_{k,\ell\leq n} \Lambda(k)\Lambda(\ell)\mathbb{E}_\Theta\left[(\widetilde{D}_{nk} - \widetilde{D}^*_{nk})(\widetilde{D}_{n\ell} - \widetilde{D}^*_{n\ell})\right].$$

We will argue as in the proof of Lemma 4.4 and thus need upper bounds for expressions like $\mathbb{E}_\Theta\left[\widetilde{D}_{nk}\widetilde{D}_{n\ell}\right]$. To study the joint behavior of $\widetilde{D}_{nk}$ and $\widetilde{D}_{n\ell}$ we compute first their generating functions.

Lemma 2. *We have for $u, v \in \mathbb{C}$*

$$\mathbb{E}_\Theta\left[u^{D_{nk}}v^{D_{n\ell}}\right] = \frac{1}{h_n}[t^n]\Big[\exp\Big(g_\Theta(t) + (u-1)(v-1)\widetilde{g}_{\Theta,\mathrm{lcm}\{k,\ell\}}(t) + (u-1)\widetilde{g}_{\Theta,k}(t) + (v-1)\widetilde{g}_{\Theta,\ell}(t)\Big)\Big] \tag{5.42}$$

with $\widetilde{g}_{\Theta,k}(t)$ as in (4.37). Furthermore, we have

$$\begin{aligned}\mathbb{E}_\Theta\left[\widetilde{D}_{nk}(\widetilde{D}_{nk}-1)\widetilde{D}_{n\ell}(\widetilde{D}_{n\ell}-1)\right] &= \frac{1}{h_n}[t^n]\left[\Big(\widetilde{g}^2_{\Theta,k}(t)\cdot\widetilde{g}^2_{\Theta,\ell}(t)\Big)\exp\big(g_\Theta(t)\big)\right]\\ &+ \frac{1}{h_n}[t^n]\left[\Big(4\widetilde{g}_{\Theta,k}(t)\cdot\widetilde{g}_{\Theta,\ell}(t)\cdot\widetilde{g}_{\Theta,\mathrm{lcm}\{k,\ell\}}(t)\Big)\exp\big(g_\Theta(t)\big)\right]\\ &+ \frac{1}{h_n}[t^n]\left[\Big(2\widetilde{g}^2_{\Theta,\mathrm{lcm}\{k,\ell\}}(t)\Big)\exp\big(g_\Theta(t)\big)\right].\end{aligned} \tag{5.43}$$

Proof. The proof of (5.42) is similar to the proof of Lemma 4.8 and we thus omit it. Equation (5.43) then follows from (5.42) by differentiating twice with respect to u and v and substituting $u = v = 1$. □

Lemma 3. *Let g_Θ belong to $\mathcal{F}(r, \vartheta, K)$. We then have for $\epsilon > 0$ arbitrary small and n large enough*

$$\begin{aligned}&\mathbb{E}_\Theta\left[\widetilde{D}_{nk}(\widetilde{D}_{nk}-1)\widetilde{D}_{n\ell}(\widetilde{D}_{n\ell}-1)\right]\\ \leq&\frac{(\vartheta+\epsilon)^4\log^4 b_n}{k^2\ell^2}+\frac{4(\vartheta+\epsilon)^3\log^3 b_n}{k\ell\operatorname{lcm}\{k,\ell\}}+\frac{2(\vartheta+\epsilon)^2\log^2 b_n}{\operatorname{lcm}\{k,\ell\}}.\end{aligned}$$

Proof. It is easy to see that there exists an $N = N(\epsilon)$ such that for $n \geq N$

$$\widetilde{g}_{\Theta,k}(t) \leq (\vartheta+\epsilon)\frac{\log b_n}{k} \text{ for } |t| \leq r(1+b_n).$$

Using Cauchys integral formula with the curve in Figure 6 then proves the lemma. □

Proof of Proposition 1. The inequality $\widetilde{D}_{nk} - \widetilde{D}^*_{nk} \leq \widetilde{D}_{nk}(\widetilde{D}_{nk}-1)$ yields

$$\begin{aligned}\mathbb{E}_\Theta\left[(\Delta_{n,\beta(n)})^2\right] &= \sum_{k,\ell=\beta(n)}^{n}\Lambda(k)\Lambda(\ell)\mathbb{E}_\Theta\left[(\widetilde{D}_{nk}-\widetilde{D}^*_{nk})(\widetilde{D}_{n\ell}-\widetilde{D}^*_{n\ell})\right]\\ &\leq \sum_{k,\ell=\beta(n)}^{n}\Lambda(k)\Lambda(\ell)\mathbb{E}_\Theta\left[\widetilde{D}_{nk}(\widetilde{D}_{nk}-1)\widetilde{D}_{n\ell}(\widetilde{D}_{n\ell}-1)\right].\end{aligned}$$

Now apply Lemma 3. Notice that $\Lambda(k)\Lambda(\ell) \neq 0$ only if $(k,\ell) = 1$ or $(k,\ell) = p$ with p a prime. We thus split the sum into the sum over all k, ℓ with $(k,\ell) = 1$ and $(k,\ell) = p$. We first consider the sum over all k, ℓ with $(k,\ell) = 1$. In this situation, we have $\operatorname{lcm}\{k,\ell\} = k\ell$. Therefore,

$$\begin{aligned}&\sum_{\substack{k,\ell=\beta(n),\\(k,\ell=1)}}^{n}\Lambda(k)\Lambda(\ell)\mathbb{E}_\Theta\left[\widetilde{D}_{nk}(\widetilde{D}_{nk}-1)\widetilde{D}_{n\ell}(\widetilde{D}_{n\ell}-1)\right]\\ \leq&\big((\vartheta+\epsilon)^4\log^4 b_n+4(\vartheta+\epsilon)^3\log^3 b_n+2(\vartheta+\epsilon)^2\log^2 b_n\big)\left(\sum_{k=\beta(n)}^{\infty}\frac{\Lambda(k)}{k^2}\right)^2\\ =&\big((\vartheta+\epsilon)^4\log^4 b_n+4(\vartheta+\epsilon)^3\log^3 b_n+2(\vartheta+\epsilon)^2\log^2 b_n\big)\left(\frac{\log(\beta(n))}{\beta(n)}\right)^2.\end{aligned}$$

By the assumptions on $\beta(n)$, this is $o(\log(n)\log\log(n))^2$. It remains to consider the case $(k, \ell) = p$. The argument for the first summand in Lemma 3 is identical as above. We now consider the second summand in the case $(k, \ell) = p$. We get

$$\sum_p \log^2 p \sum_{\substack{a,b\geq 1 \\ p^a, p^b \geq \beta(n)}} \frac{1}{p^{a+b+\max\{a,b\}}} \leq \sum_p \log^2 p \sum_{\substack{a,b\geq 1 \\ p^a, p^b \geq \beta(n)}} \frac{1}{p^{\frac{3}{2}a+\frac{3}{2}b}}$$
$$= \left(\sum_{k\geq\beta(n)} \frac{\Lambda(k)}{k^{\frac{3}{2}}}\right)^2 \leq \left(\sum_{k\geq\beta(n)} \frac{\log k}{k^{\frac{3}{2}}}\right)^2 = \frac{\log^2 \beta(n)}{\beta(n)}.$$

This gives again $o(\log(n)\log\log(n))^2$. The argument for the third summand is similar and also gives $o(\log(n)\log\log(n))^2$. This completes the proof. $\square$

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